

Harmonicity in $\mathcal{N} = 4$ supersymmetry and its quantum anomaly

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Abstract

The holomorphicity property of $\mathcal{N} = 1$ superpotentials or of $\mathcal{N} = 2$ F-terms involving vector multiplets is generalized to the case of $\mathcal{N} = 4$ 1/2-BPS effective operators defined in harmonic superspace. The resulting *harmonicity* equations are shown to control the moduli dependence of the couplings of higher dimensional operators involving powers of the $\mathcal{N} = 4$ Weyl superfield, computed by $\mathcal{N} = 4$ topological amplitudes. These equations can also be derived on the string side, exhibiting an anomaly from world-sheet boundary contributions that leads to recursion relations for the *non-analytic* part of the couplings.

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1 Introduction

An important property of $\mathcal{N} = 2$ F-terms involving vector multiplets is holomorphicity, implying that the corresponding couplings are holomorphic functions of the vector moduli. This applies, for instance, to the couplings F_g of the higher dimensional F-terms W^{2g} , where W is the self-dual (chiral) Weyl superfield, appearing in the string effective action [1]. On the other hand, the couplings F_g are computed by the topological partition function of the $\mathcal{N} = 2$ twisted Calabi-Yau σ -model associated to the six-dimensional compactification manifold of type II string theory in four dimensions [2, 1]. It turns out, however, that there is a holomorphic anomaly, related to a violation of the conservation of the BRST current in the topological theory, implying that antichiral fields do not decouple at the quantum level [3, 4, 2]. The anomaly arises from boundary contributions and takes the form of an equation that amounts to a recursion relation for the non-holomorphic part of the couplings F_g . From the point of view of the string effective action, it arises from the quantum integration over massless states that is unavoidable when computing on-shell physical amplitudes [1].

These couplings were generalized recently to 1/2-BPS terms of $\mathcal{N} = 4$ compactifications, involving powers of the (superdescendant of the) $\mathcal{N} = 4$ chiral Weyl superfield $K^{++} = D_- D_- W$, where D_- are particular $SU(4)$ projections of the spinor derivatives. We recall that the $\mathcal{N} = 4$ gravity multiplet contains, besides the graviton and the four gravitini, six graviphotons, one complex graviscalar and four spin-1/2 Weyl fermions [5]. Moreover, there is an $SU(4)$ R-symmetry, transforming the gravitini in the fundamental and the graviphotons in the vector representation. The (linearized on-shell) superfield K^{++} satisfies 1/2-BPS shortening conditions. Its lowest component is the (self-dual) graviphoton field strength and its next bosonic components are the (self-dual) Riemann tensor and the second derivative of the graviscalar. Another basic 1/2-BPS superfield in the $\mathcal{N} = 4$ theory is the (linearized on-shell) vector multiplet Y^{++} . Its lowest component are the scalar moduli transforming in the vector representation of $SU(4)$, like the graviphoton field strengths.

In [5] two series of 1/2-BPS couplings were found: $\mathcal{F}_g^{(1)} \bar{K}^2 K^{2g}$ and $\mathcal{F}_g^{(3)} K^{2(g+1)}$. Here $\mathcal{F}_g^{(1)}$ and $\mathcal{F}_g^{(3)}$ are functions of the $\mathcal{N} = 4$ moduli vector multiplets Y^{++} and of the $SU(4)$ harmonic variables that can again be computed by topological amplitudes on $K3 \times T^2$ of genus g and $g+1$, respectively. Actually, in six dimensions there is also the series $F_g^{(6d)} W_{6d}^{4g}$, where W_{6d} is a similar Weyl superfield of the six-dimensional gravity multiplet and $F_g^{(6d)}$ is given by a topological theory on $K3$ [6].

In this work, we study the question of what is the analog of $\mathcal{N} = 2$ holomorphicity for such 1/2-BPS $\mathcal{N} = 4$ couplings. The main novelty of the generalization is that the relevant R-symmetry group becomes non-Abelian, transforming non-trivially the superfields K^{++} and Y^{++} . As a consequence, the notion of chirality of the $\mathcal{N} = 2$ theory is replaced by Grassmann analyticity (or 1/2-BPS shortness). The natural framework for studying this problem and covariantizing the expressions is then harmonic superspace [7, 8, 9]. By introducing $SU(4)$ harmonic variables one can define K^{++} as a particular harmonic projection of the sextet of superfields $K_{ij} = -K_{ji} = D_i D_j W$, associated to a corresponding

1/2-BPS subspace of the full $\mathcal{N} = 4$ superspace. Supersymmetry then implies that the coupling coefficients \mathcal{F}_g are functions of the same harmonic projected vector superfields Y^{++} living in the same 1/2-BPS subspace. Thus, $\mathcal{F}_g(Y^{++})$ is independent of the five remaining projections of the sixplet of the scalar moduli. This defines a notion of analyticity that naturally generalizes $\mathcal{N} = 2$ holomorphicity for the chiral $\mathcal{N} = 2$ vector multiplets.

In this work, we show that the above property of analyticity can be formulated in terms of a set of differential constraints on the couplings \mathcal{F}_g of $\mathcal{N} = 4$ 1/2-BPS effective operators. They express the property of the analytic functions $\mathcal{F}_g(Y^{++})$ that, when expanded in powers of the harmonic variables and the scalar fields, the coefficients should form symmetric traceless tensors of $SO(6)$. This yields two non-trivial equations. The first requires one scalar and one harmonic derivative to vanish and coincides with the so-called ‘harmonicity’ equation found previously in string computations, up to an anomaly [6, 10, 5]. The second involves two scalar derivatives and gets modified in supergravity by an additive constant term due to the curvature of the scalar kinetic terms. Both equations are checked by an explicit string computation for $\mathcal{F}_g^{(3)}$, which receives one-loop corrections on the heterotic side for all g , and are found to be corrected by anomaly terms due to world-sheet boundary contributions that spoil the naive expectation of analyticity, in a way similar to the holomorphic anomaly equation of the $\mathcal{N} = 2$ F_g ’s. The resulting equations are reduced again to recursion relations for the non-analytic part of the moduli-dependent couplings.

We finally extend the above results to six-dimensional $\mathcal{N} = (1, 1)$ supersymmetry, where the R-symmetry group is $SO(4)$. In particular, we consider the decompactification limit of $\mathcal{F}_g^{(3)}$ that gives rise to a new six-dimensional series of 1/2-BPS couplings of the form $\mathcal{F}_g^{\text{dec}} W_{6d}^{2(g+1)}$. These couplings, although not exactly topological in six dimensions (the space-time part is not decoupled), become topological upon compactification to four dimensions on a two-torus. Despite this fact, $\mathcal{F}_g^{\text{dec}}$ satisfy the same analyticity condition as $F_g^{(6d)}$ of W_{6d}^{4g} since they are both 1/2-BPS. We then derive the corresponding analyticity equations, together with the anomaly terms.

The paper is organized as follows. In Section 2, we describe the analyticity conditions of the 1/2-BPS couplings in the case of global $\mathcal{N} = 4$ supersymmetry. We first introduce the $SU(4)$ harmonic variables and the harmonic projected vector and Weyl superfields, Y^{++} and K^{++} respectively. We then derive the differential equations for the couplings \mathcal{F}_g of the higher-derivative operators involving powers of $(K^{++})^2$, as described above. In Section 3, we study the effects of the curvature of the scalar manifold that parametrizes the coset $SO(6, n)/SO(6) \times SO(n)$, where n is the number of vector multiplets. We show in particular that the second-order derivative equation in the scalar fields gets modified by an additional term proportional to the Weyl weight of the operator. In Section 4, we go to curved superspace in the framework of $\mathcal{N} = 4$ conformal supergravity and derive the fully covariantized final expressions of the higher-derivative couplings. In Section 5, we give a brief review of the $\mathcal{N} = 4$ topological amplitudes in string theory and recall the expression for $\mathcal{F}_g^{(3)}$ obtained from a one-loop string computation on the heterotic side. In Section 6, we derive the harmonicity relation which is first order in the scalar field derivatives, exhibiting a boundary anomaly that invalidates the expected vanishing result.

In Section 7, we obtain the second-order constraint which is also modified by an anomaly. In Section 8, we generalize our analysis to six dimensions. We first introduce the $SO(4)$ harmonic variables and derive the harmonicity equations for the couplings of the 1/2-BPS terms. We then consider the decompactification limit of $\mathcal{F}_g^{(3)}$ and compute the two analyticity equations modified by the anomalous terms. Finally, Section 9 contains some concluding remarks.

2 Global $\mathcal{N} = 4$ supersymmetry

2.1 $SU(4)$ harmonic variables

We consider $\mathcal{N} = 4$ supersymmetry in four dimensions whose automorphism group is $SU(4)$. We introduce harmonic variables [7, 8, 9] on the coset $SU(4)/S(U(2) \times U(2))$ in the form of matrices $(u_i^{+a}, u_i^{-\dot{a}}) \in SU(4)$. They have an index $i = 1 \dots 4$ transforming under the fundamental irrep of $SU(4)$ and indices $a, \dot{a} = 1, 2$ of $SU(2) \times SU(2)$ as well as $U(1)$ charges ± 1 . Together with their complex conjugates $\bar{u}_{+a}^i = \overline{(u_i^{+a})}$, $\bar{u}_{-\dot{a}}^i = \overline{(u_i^{-\dot{a}})}$ they satisfy the unitarity conditions

$$\begin{aligned} u_i^{+a} \bar{u}_{+b}^i &= \delta_b^a, & u_i^{-\dot{a}} \bar{u}_{-\dot{b}}^i &= \delta_{\dot{b}}^{\dot{a}}, & u_i^{+a} \bar{u}_{-\dot{b}}^i &= u_i^{-\dot{a}} \bar{u}_{+b}^i = 0 \\ u_i^{+a} \bar{u}_{+a}^j + u_i^{-\dot{a}} \bar{u}_{-\dot{a}}^j &= \delta_i^j \end{aligned} \quad (2.1)$$

and the unit determinant condition

$$\epsilon^{ijkl} u_i^{+a} u_j^{+b} u_k^{-\dot{a}} u_l^{-\dot{b}} = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \quad (2.2)$$

(with $\epsilon^{1234} = \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{12} = -\epsilon_{\dot{1}\dot{2}} = 1$).

The harmonic functions have harmonic expansions homogeneous under the action of the subgroup $S(U(2) \times U(2))$. The harmonic expansions are organized in irreps of $SU(4)$, keeping the balance of projected indices so that the overall $SU(2) \times SU(2)$ indices and the $U(1)$ charge are always the same. In what follows we shall often make use of functions depending on vector-like combinations of $SU(4)$ harmonics (i.e., with harmonics on $SO(6)/SO(4) \times SO(2)$) of the type $u_{ij}^M = -u_{ji}^M$, $M = (++, --, a\dot{a})$ (and their conjugates $\bar{u}_M^{ij} = \overline{u_{ij}^M}$)

$$\begin{aligned} u_{ij}^{++} &= u_i^{+a} \epsilon_{ab} u_j^{+b} \\ u_{ij}^{--} &= u_i^{-\dot{a}} \epsilon_{\dot{a}\dot{b}} u_j^{-\dot{b}} \\ u_{ij}^{a\dot{a}} &= u_{[i}^{+a} u_{j]}^{-\dot{a}}, \end{aligned} \quad (2.3)$$

where $[ij]$ denotes weighted antisymmetrization. They form $SO(6)$ matrices $u_N^M = \Gamma_N^{ij} u_{ij}^M$ where Γ^M are the gamma matrices of $SO(6)$. The vector-like harmonics satisfy algebraic

conditions expressing the fact that $u_N^M \in SO(6)$ and following from the conditions on the underlying $SU(4)$ harmonics:

$$\begin{aligned}
u_{ij}^{++} \epsilon^{ijkl} u_{kl}^{--} &= 4 \\
u_{ij}^{a\dot{a}} \epsilon^{ijkl} u_{kl}^{b\dot{b}} &= \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \\
u_{ij}^{++} \epsilon^{ijkl} u_{kl}^{++} &= u_{ij}^{--} \epsilon^{ijkl} u_{kl}^{--} = u_{ij}^{++} \epsilon^{ijkl} u_{kl}^{a\dot{a}} = u_{ij}^{--} \epsilon^{ijkl} u_{kl}^{a\dot{a}} = 0 \\
u_{ij}^{++} u_{kl}^{--} + u_{ij}^{--} u_{kl}^{++} - 2 u_{ij}^{a\dot{a}} \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} u_{kl}^{b\dot{b}} &= \epsilon_{ijkl}
\end{aligned} \tag{2.4}$$

An example of a harmonic function which we shall frequently encounter is $\phi^{++}(u) = \phi^{ij} u_{ij}^{++} + \phi_{mn}^{ijkl} u_{ij}^{++} u_{kl}^{++} u_{++}^{mn} + \dots$. The first component in this expansion is a $\underline{6}$ of $SU(4)$ (or a vector of $SO(6)$) $\phi^{ij} = -\phi^{ji}$. The higher components give rise to higher-dimensional irreps, but we shall not need them here.

The harmonic derivatives can be viewed as the covariant derivatives on the harmonic coset $SU(4)/S(U(2) \times U(2))$, or equivalently, as the generators of the $SU(4)$ algebra written down in an $S(U(2) \times U(2))$ basis (see Section 3). This means that they are invariant under the left action of the group $SU(4)$, but covariant under the right action of the subgroup $S(U(2) \times U(2))$. They can be split into generators of the subalgebra $S(U(2) \times U(2))$:

$$\begin{aligned}
D_{+a}^{+b} &= \left(u_i^{+b} \frac{\partial}{\partial u_{i+}^{+a}} - \bar{u}_{+a}^i \frac{\partial}{\partial \bar{u}_{+b}^i} \right) - \text{trace} \\
D_{-\dot{a}}^{-\dot{b}} &= \left(u_i^{-\dot{b}} \frac{\partial}{\partial u_{i-}^{-\dot{a}}} - \bar{u}_{-\dot{a}}^i \frac{\partial}{\partial \bar{u}_{-\dot{b}}^i} \right) - \text{trace} \\
D_0 &= \left(u_i^{+a} \frac{\partial}{\partial u_{i+}^{+a}} - \bar{u}_{+a}^i \frac{\partial}{\partial \bar{u}_{+a}^i} \right) - \left(u_i^{-\dot{a}} \frac{\partial}{\partial u_{i-}^{-\dot{a}}} - \bar{u}_{-\dot{a}}^i \frac{\partial}{\partial \bar{u}_{-\dot{a}}^i} \right)
\end{aligned} \tag{2.5}$$

and of the coset:

$$\begin{aligned}
D_{+a}^{-\dot{b}} &= u_i^{-\dot{b}} \frac{\partial}{\partial u_{i+}^{+a}} - \bar{u}_{+a}^i \frac{\partial}{\partial \bar{u}_{-\dot{b}}^i} \\
D_{-\dot{a}}^{+b} &= u_i^{+b} \frac{\partial}{\partial u_{i-}^{-\dot{a}}} - \bar{u}_{-\dot{a}}^i \frac{\partial}{\partial \bar{u}_{+b}^i} .
\end{aligned} \tag{2.6}$$

The harmonic derivatives are differential operators preserving the defining algebraic constraints (2.1), (2.2).

The derivatives (2.5) act homogeneously on the harmonic functions. For instance, the function $\phi^{++}(u)$ above has no $SU(2) \times SU(2)$ indices, but has $U(1)$ charge, hence

$$D_{+a}^{+b} \phi^{++}(u) = D_{-\dot{a}}^{-\dot{b}} \phi^{++}(u) = 0, \quad D_0 \phi^{++}(u) = 2 \phi^{++}(u) . \tag{2.7}$$

The harmonic expansion of this function defines an infinitely reducible representation of $SU(4)$. It can be made irreducible by requiring that the raising operator $D_{-\dot{a}}^{+b}$ annihilate the function:

$$D_{-\dot{a}}^{+b} \phi^{++}(u) = 0 \quad \Rightarrow \quad \phi^{++}(u) = \phi^{ij} u_{ij}^{++} . \tag{2.8}$$

In other words, such a function is a highest-weight state of the $\underline{6}$ of $SU(4)$. The irreducibility condition (2.8) is also called a condition for harmonic (H-) analyticity.

2.2 Grassmann analytic on-shell superfields

The introduction of harmonic variables allows us to define ‘1/2 BPS short’ or Grassmann (G-) analytic¹ superfields.² They depend only on half of the Grassmann variables which can be chosen to be $\theta_\alpha^{+a} = \theta_\alpha^i u_i^{+a}$ and $\bar{\theta}_{-\dot{a}} = \bar{u}_{-\dot{a}}^i \bar{\theta}_i^{\dot{a}}$. One such superfield is the *linearized on-shell* vector multiplet

$$\begin{aligned} Y^{++}(x^\mu, \theta^+, \bar{\theta}_-, u) &= \phi^{ij} u_{ij}^{++} + \theta_\alpha^{+a} \epsilon_{ab} \psi^{\alpha i} u_i^{+b} + \bar{u}_{-\dot{a}}^i \bar{\psi}_{\dot{\alpha} i} \epsilon^{\dot{a} b} \bar{\theta}_{-\dot{b}}^{\dot{\alpha}} \\ &+ \theta^{+a} \sigma^{\mu\nu} \theta^{+b} \epsilon_{ab} F_{(+)\mu\nu} + \bar{\theta}_{-\dot{a}} \tilde{\sigma}^{\mu\nu} \bar{\theta}_{-\dot{b}} \epsilon^{\dot{a} b} F_{(-)\mu\nu} + \text{derivative terms} . \end{aligned} \quad (2.9)$$

Here $\phi^{ij} = \frac{1}{2} \epsilon^{ijkl} \bar{\phi}_{kl}$ are the six real scalars, ψ_i^α are the four Majorana gluinos and $F_{(\pm)\mu\nu}$ is the (anti)self-dual part of the gluon field strength. To exhibit manifest G-analyticity, one has to choose the appropriate analytic basis in superspace,

$$x^\mu \rightarrow x^\mu + i\theta^{+a} \sigma^\mu \bar{\theta}_{+a} + i\theta^{-\dot{a}} \sigma^\mu \bar{\theta}_{-\dot{a}} , \quad (2.10)$$

analogous to the familiar chiral basis. Note that the harmonic dependence here is cut down to linear in the vector-like and fundamental harmonics. This is typical for on-shell multiplets which, in addition to the G-analyticity condition, also satisfy the H-analyticity condition

$$D_{-\dot{a}}^{+b} Y^{++}(\theta^+, \bar{\theta}_-, u) = 0 . \quad (2.11)$$

Here the harmonic derivative is supersymmetrized by going to the manifestly G-analytic superspace coordinates (2.10). One can show that the ‘ultrashort’ on-shell superfield (2.9) is the solution to the simultaneous conditions for G- and H-analyticity [9, 12].

Another example of a G-analytic superfield is the *linearized* on-shell Weyl multiplet. It is obtained from the off-shell chiral Weyl superfield [14]

$$W(\theta_\alpha^i) = \Phi + \theta_\alpha^i \theta_\beta^j (\sigma_{\mu\nu}^{\alpha\beta} T_{(+)[ij]}^{\mu\nu} + \epsilon^{\alpha\beta} S_{(ij)}) + \dots . \quad (2.12)$$

Here Φ is the physical scalar and $T_{(+)}$ is the self-dual part of the sixplet of graviphoton field strengths, while S is an auxiliary field. On shell the latter must vanish, hence the additional constraint

$$\epsilon_{\alpha\beta} D_i^\alpha D_j^\beta W = 0 . \quad (2.13)$$

Now, define the superfield (a superdescendant of W)

$$K_{\mu\nu}^{++} = (\sigma_{\mu\nu})_{\alpha\beta} D_{-\dot{a}}^\alpha D_{-\dot{b}}^\beta \epsilon^{\dot{a}\dot{b}} W \quad (2.14)$$

where we have projected the $SU(4)$ indices of $D_i^\alpha D_j^\beta$ with the harmonics $\bar{u}_{-\dot{a}}^i \bar{u}_{-\dot{b}}^j$. This superfield is annihilated by half of the spinor derivatives and hence is 1/2 BPS short.

¹A more systematic derivation of the G-analytic superfields as functions on a coset of the $\mathcal{N} = 4$ superconformal algebra $PSU(2, 2/4)$ will be given in Section 4.

²The notion of Grassmann analyticity (with breaking of the R symmetry) was first proposed in [11] in the context of the $\mathcal{N} = 2$ hypermultiplet. Later on it was made R-symmetry covariant in the framework of $\mathcal{N} = 2$ harmonic superspace in [7]. The harmonic superspace description of the $\mathcal{N} = 3$ *off-shell* vector multiplet was given in [8], and that of the $\mathcal{N} = 4$ *on-shell* vector multiplet in [9].

Indeed, this is true for the projections $\bar{D}_{\dot{\alpha}}^{+a}$ since $\{\bar{D}^{+a}, D_{-b}\} = 0$ and $\bar{D}_{\dot{\alpha}}^i W = 0$ (chirality). Further, hitting (2.14) with $D_{-\dot{a}}^\gamma$ we obtain zero as a consequence of the projection of the on-shell constraint (2.13) with $\bar{u}_{-\dot{a}}^i \bar{u}_{-b}^j$. We conclude that $K_{\mu\nu}^{++}$ satisfies the G-analyticity constraints

$$D_{-\dot{a}}^\alpha K_{\mu\nu}^{++} = \bar{D}_{\dot{\alpha}}^{+a} K_{\mu\nu}^{++} = 0 \quad (2.15)$$

which imply that it depends on half of the θ 's (bosons only):

$$K_{\mu\nu}^{++}(\theta^+, \bar{\theta}_-, u) = T_{(+)\mu\nu}^{ij} u_{ij}^{++} + \theta^{+a} \sigma^{\lambda\rho} \theta^{+b} \epsilon_{ab} R_{(+)\mu\nu\lambda\rho} + \bar{\theta}_{-\dot{a}} \tilde{\sigma}^\lambda \sigma_{\mu\nu} \sigma^\rho \bar{\theta}_{-\dot{b}} \epsilon^{\dot{a}\dot{b}} \partial_\lambda \partial_\rho \Phi + \dots \quad (2.16)$$

In addition, the harmonic dependence of $K_{\mu\nu}^{++}$ is restricted to be linear. As in (2.11), this follows from the condition for H-analyticity

$$D_{-\dot{a}}^{+b} K_{\mu\nu}^{++} = 0 \quad (2.17)$$

This is another example of an ultrashort superfield. Note, however, that it is not a primary object but rather a superdescendant of the chiral on-shell Weyl multiplet.

Repeating the same steps, but this time starting with the antichiral superfield $\bar{W}(\bar{\theta})$ we obtain the other half of the on-shell Weyl multiplet. It is again described by an ultrashort superfield of the same type,

$$\bar{K}_{\mu\nu}^{++}(\theta^+, \bar{\theta}_-, u) = T_{(-)\mu\nu}^{ij} u_{ij}^{++} + \bar{\theta}_{-\dot{a}} \tilde{\sigma}^{\lambda\rho} \bar{\theta}_{-\dot{b}} \epsilon^{\dot{a}\dot{b}} R_{(-)\mu\nu\lambda\rho} + \theta^{+a} \sigma^\lambda \tilde{\sigma}_{\mu\nu} \tilde{\sigma}^\rho \theta^{+b} \epsilon_{ab} \partial_\lambda \partial_\rho \bar{\Phi} + \dots \quad (2.18)$$

Note that in the $\mathcal{N} = 4$ G-analytic superspace there exists a special conjugation \sim combining complex conjugation with a reflection on the harmonic coset, such that G-analyticity is preserved. In this sense $Y^{++} = \widetilde{Y^{++}}$ and $\bar{K}^{++} = \widetilde{\bar{K}^{++}}$, which implies, in particular, the reality condition on the six scalars in Y .

2.3 Higher-derivative couplings

After having defined the G-analytic superfields (2.9) and (2.16), we now want to construct the corresponding effective action couplings. Recently, by studying special higher loop scattering processes in the gravitational sector of type II superstring theory compactified on $K3 \times T^2$ (or the corresponding dual formulation of heterotic string on T^6 , as we will review in Section 5), the following two terms were found in [5]:

$$S_1 = \int d^4x \, du \, d^4\theta^+ d^4\bar{\theta}_- (\bar{K}_{\mu\nu}^{++} \bar{K}^{++\mu\nu}) (K_{\rho\sigma}^{++} K^{++\rho\sigma})^g F_1^{-4(g-1)}(Y_A^{++}, u) \quad (2.19)$$

$$S_2 = \int d^4x \, du \, d^4\theta^+ d^4\bar{\theta}_- (K_{\mu\nu}^{++} K^{++\mu\nu})^{g+1} F_2^{-4(g-1)}(Y_A^{++}, u) \quad (2.20)$$

where $A = 1 \dots n$ is an $SO(n)$ vector index labeling the coordinates of the coset of physical scalars (see Section 3). In fact, if considered as g - and $(g+1)$ -loop contributions respectively, both of these terms lead to so-called *topological amplitudes*, that is the corresponding physical amplitudes are computed by correlation functions of the $\mathcal{N} = 4$ topological string

on $K3 \times T^2$. However, unlike the $\mathcal{N} = 2$ case (see [1]) these correlation functions are not simply the topological partition function, but differ from it by additional operator insertions in the twisted version of the theory. Actually, for convenience and notational simplicity, we changed the notation of the two couplings from [5]: $F_1^{-4(g-1)}$ corresponds to $\mathcal{F}_g^{(1)}$, while $F_2^{-4(g-1)}$ corresponds to $\mathcal{F}_g^{(3)}$, with the upper index denoting the $U(1)$ charge.

It is important to stress upon two points concerning the effective action terms (2.19) and (2.20):

1. The Grassmann measure is G-analytic, i.e. it involves only half of the projected θ 's, and so must be the integrand, otherwise supersymmetry will be broken. This is why we have to use the *linearized on-shell* superfields Y^{++} and K^{++} which are G-analytic like the measure.
2. The harmonic integral should produce an $SU(4)$ invariant, i.e. it picks out the $SU(4)$ singlet part of the integrand. This is only possible if the latter is a *chargeless* harmonic function. For example, $f(u) = f_0 + f_{kl}^{ij} u_{ij}^{++} u_{++}^{kl} + \dots$ integrates to $\int du f(u) = f_0$, but a charged function like $f^{++} = f^{ij} u_{ij}^{++} + \dots$ will have a vanishing integral. Notice that for this reason the harmonic integral should always be done last, after the Grassmann integrals, since the latter are charged.

In our case (2.19), (2.20) the functions $F_{1,2}$ carry $U(1)$ charge $-4(g-1)$ needed to compensate that of the factor K ($+4(g+1)$) and of the Grassmann measure (-8). Given the fact that the argument Y^{++} of F has a positive charge, we have to introduce a set of *constant* $SU(4)$ multispinors

$$\xi^{-2p}(u) \equiv \xi_{(i_1 \dots i_p)(j_1 \dots j_p)} \bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_p j_p} + \dots \quad (2.21)$$

thus explicitly breaking $SU(4)$.³ The dots denote higher-order terms in the harmonic expansion of the coefficients $\xi(u)$ which will not be of interest for us, see below. Note that the product of vector-like harmonics forms an irreducible representation of $SO(6)$, a symmetric traceless tensor of rank p (recall that $(\bar{u}_{++})^2 = 0$, see (2.4)). In $SU(4)$ notation this means that the indices i and j of the coefficients ξ are separately symmetrized, but antisymmetrized between i and j , i.e. we are dealing with the irrep $(0p0)$. In what follows this fact will be of crucial importance. So, we consider the potential ($m = 2(g-1)$; the $SO(n)$ index A and the labels 1, 2 are suppressed)

$$F^{-2m}(Y^{++}, u) = \sum_{n=0}^{\infty} \xi^{-2(m+n)}(u) (Y^{++})^n. \quad (2.22)$$

The factors K in (2.19) and (2.20) contribute, among others, the terms

$$(\theta^+)^4 (\bar{\theta}_-)^4 R_{(+)}^2 R_{(-)}^2 (T_{(+)}^{++})^m, \quad (\theta^+)^4 (\bar{\theta}_-)^4 R_{(+)}^2 (\partial \partial \Phi)^2 (T_{(+)}^{++})^m, \quad (2.23)$$

³The other possibility, which we do not consider here, would be to use singular functions involving inverse powers of fields.

respectively. The θ 's saturate the superspace measure and are integrated out. The remainder has a harmonic charge,

$$(T^{++})^m = T^{(i_1(j_1) \dots T^{i_m)j_m)} u_{i_1 j_1}^{++} \dots u_{i_m j_m}^{++} \quad (2.24)$$

which is compensated by the factor F in order to have a non-vanishing harmonic integral (i.e., an $SU(4)$ singlet). Clearly, (2.24) is an irrep of $SO(6)$, a symmetric traceless tensor of rank m . This can be reformulated as the highest-weight condition (cf. (2.8))

$$D_{-\dot{a}}^{+b}(T^{++})^m = 0 . \quad (2.25)$$

A similar condition holds for the entire effective action expressions (2.19) and (2.20) of the graviphoton field strength superfield.

The singlet needed for the harmonic integral is obtained by combining (2.24) with the matching irrep in F . Consider the harmonic structure of F (all $\theta = 0$):

$$\begin{aligned} F^{-2m}(\phi^{++}, u) &= \sum_{n=0}^{\infty} \xi_{(i_1 \dots i_{m+n})(j_1 \dots j_{m+n})} \bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_{m+n} j_{m+n}} \\ &\times \phi^{(k_1 l_1} \dots \phi^{k_n) l_n)} u_{k_1 l_1}^{++} \dots u_{k_n l_n}^{++} . \end{aligned} \quad (2.26)$$

Here we have restricted the harmonic expansion (2.21) of the coefficient function $\xi^{-2(m+n)}(u)$ to the lowest-rank $SO(6)$ irrep. The higher-rank terms are irrelevant due to the gauge invariance of the couplings (2.19), (2.20). Indeed, consider adding a total supersymmetrized harmonic derivative $D_{-\dot{a}}^{+b} \Lambda^{(-4g+2)\dot{a}}_{\dot{b}}(\theta^+, \bar{\theta}_-, u)$ to the potential $F^{-4(g-1)}$. After integration by parts (the G-analytic measure allows this), $D_{-\dot{a}}^{+b}$ annihilates the on-shell superfield K^{++} (recall (2.17)), hence the gauge invariance of (2.19), (2.20) with the G-analytic parameter Λ . By examining the harmonic expansion of $\Lambda(0, 0, u)$ one can show that all the omitted terms in (2.26) can be gauged away.

The gauge-fixed function (2.26) satisfies two differential conditions. The first one expresses the fact that it is a function only of the projection ϕ^{++} of the $SO(6)$ vector of physical scalars:

$$\frac{\partial}{\partial \phi^{--}} F^{-2m} = \frac{\partial}{\partial \phi^{a\dot{a}}} F^{-2m} = 0 . \quad (2.27)$$

This is yet another kind of analyticity condition (S-analyticity), this time with respect to the scalars (which in fact are the coordinates on the curved manifold $SO(6, n)/SO(6) \times SO(n)$, see Section 3). The second one restricts the harmonic dependence

$$D_{+a}^{-\dot{b}} F^{-2m} = \epsilon_{ab} \phi^{b\dot{b}} \frac{\partial}{\partial \phi^{++}} F^{-2m} . \quad (2.28)$$

Note that if the right-hand side in (2.28) vanished, this would be a condition defining a lowest-weight state of $SU(4)$ (or an $SO(6)$ tensor of rank m). However, the dependence on the scalars makes the harmonic structure in (2.26) reducible.

From (2.26) we have to extract the irreducible harmonic structure $\bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_m j_m}$ needed to match the conjugate structure in (2.24). It is obtained by contracting all the u^{++} in (2.26) with a subset of the \bar{u}_{++} , using $\bar{u}_{++}^{ij} u_{kl}^{++} = -1/3 \delta_k^{[i} \delta_l^{j]} + \dots$ (see (2.1)). This confirms that the omitted terms in the harmonic expansion of ξ in (2.26) cannot contribute - they contain higher-rank $SO(6)$ tensors. The result is the *relevant part* of the function F , or the *reduced function*

$$\mathcal{F}^{-2m} = \sum_n \xi_{(i_1 \dots i_{m+n})(j_1 \dots j_{m+n})} \bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_m j_m} \phi^{i_{m+1} j_{m+1}} \dots \phi^{i_{m+n} j_{m+n}} . \quad (2.29)$$

Notice the full symmetrization of the i and j indices of the ξ tensor inherited from (2.26). As required, the reduced function is manifestly H-analytic (i.e., $SU(4)$ irreducible),

$$D_{+a}^{-b} \mathcal{F}^{-2m} = 0 . \quad (2.30)$$

However, now the manifest S-analyticity (i.e., the dependence only on ϕ^{++}) of (2.26) is lost.

It should be made clear that (2.29) is just a rearrangement of the harmonic expansion of the gauge-fixed function F^{-2m} . The information contained in this function is encoded in the fact that the coefficients $\xi_{(i_1 \dots i_{m+n})(j_1 \dots j_{m+n})}$, which are the same in (2.26) and (2.29), form the $SU(4)$ irrep $(0, m+n, 0)$ ⁴. This information can be translated into two types of differential constraints on the function \mathcal{F} . In general, the harmonic and scalar factors in (2.29) form the reducible representation $(0m0) \otimes \prod_{p=1}^n (010) \rightarrow (0, m+n, 0) + \dots$. The relevant projection $(0, m+n, 0)$ is obtained by symmetrizing all the i and separately all the j indices (the antisymmetry of the i 's with the j 's is automatic). Any other irrep in this tensor product will have a subset of the i 's (and of the j 's) antisymmetrized. The product of two u 's is irreducible, $(010) \otimes (010) \rightarrow (020)$ as follows from the commuting nature of the $SU(4)$ harmonics \bar{u}_{+a}^i . The antisymmetrization of indices carried by the \bar{u} 's and the ϕ 's is ruled out by the so-called *harmonicity* condition:

$$\epsilon^{pqrs} \frac{\partial}{\partial \bar{u}_{+a}^q} \frac{\partial}{\partial \phi_A^{rs}} \mathcal{F} = 0 , \quad (2.31)$$

where we have restored the $SO(n)$ index A and suppressed the $U(1)$ charge superscript $-2m$. The above equation forbids the decomposition $(100) \otimes (010) \rightarrow (001)$. This constraint involves partial derivatives with respect to \bar{u}_+ . Strictly speaking, such an operation is illegal in the harmonic formalism, since the variables u are not independent, as can be seen from (2.1), (2.2). However the above equation can be rewritten using covariant harmonic derivatives introduced in (2.5) and (2.6) as

$$\epsilon^{pqrs} \left(u_q^{+b} D_{+b}^{+a} + \frac{1}{2} u_q^{+a} D_0 + \bar{u}_q^{-b} D_{-b}^{+a} \right) \frac{\partial}{\partial \phi_A^{rs}} \mathcal{F} = 0 . \quad (2.32)$$

Indeed, it is easy to see that this equation reduces to (2.31) since our function \mathcal{F} explicitly involves only \bar{u}_+ harmonics. The D_0 term in (2.32) is just to remove the contribution from

⁴ $SU(4)$ irreps with Dynkin labels $(0p0)$ are equivalent to rank p symmetric traceless tensor of $SO(6)$.

the trace parts in D_{+b}^{+a} as defined in (2.5) which measures the total $U(1)$ charge $-2m$ of \mathcal{F} . In the following however we will continue to write the formula using partial derivatives with respect to \bar{u}_+ .

Further, the antisymmetrization of indices carried by the ϕ 's is ruled out by the constraint

$$\epsilon^{pqrs} \frac{\partial}{\partial \phi_A^{pq}} \frac{\partial}{\partial \phi_B^{rt}} \mathcal{F} = 0 \quad (2.33)$$

which forbids the decomposition $(010) \otimes (010) \rightarrow (101) \oplus (000)$. In $SO(6)$ (vector) notation (2.33) reads

$$\frac{\partial}{\partial \phi_A^{[M}} \frac{\partial}{\partial \phi_B^{N]}} \mathcal{F} = 0 , \quad (2.34)$$

$$\frac{\partial}{\partial \phi_A^M} \frac{\partial}{\partial \phi_B^M} \mathcal{F} = 0 . \quad (2.35)$$

Here we do not take into account the fact that the physical scalars ϕ parametrize a curved manifold and hence the derivatives in (2.33) should be considered covariant with respect to the metric of the manifold. In Section 3 we show that this leads to a modification of (2.35) by a term proportional to δ_{AB} .

3 The coset of physical scalars

3.1 The coset $SO(6, n)/SO(6) \times SO(n)$

Here we briefly recall why the scalars of $\mathcal{N} = 4$ Poincaré supergravity describe the coset space $SO(6, n)/SO(6) \times SO(n)$ [13, 14, 15].

$\mathcal{N} = 4$ Poincaré supergravity is obtained by coupling the *off-shell* Weyl multiplet to $6 + n$ free vector multiplets. The first six are compensating multiplets (i.e. their kinetic terms have the wrong sign), the remaining n are physical. Each vector multiplet supplies 6 scalars, so the total number is $6(6 + n)$. We denote the first 6×6 by φ_N^M and the other $6 \times n$ by ϕ_A^M , where $M, N = 1 \dots 6$ and $A = 1 \dots n$ are $SO(6)$ and $SO(n)$ vector indices, respectively.

The Weyl multiplet contains an auxiliary field $D_{MN} = D_{NM}$, $D_{MM} = 0$ in the **20'** of $SO(6) \sim SU(4)$. It serves as a Lagrange multiplier for the following quadratic combination of scalars:

$$D_{MN} (\varphi_K^M \varphi_K^N - \phi_A^M \phi_A^N) . \quad (3.1)$$

So, it imposes an algebraic constraint which eliminates 20 of the scalars. In addition, one makes a Weyl (dilatation) gauge choice for the trace of the quadratic form in (3.1), thus fixing yet another scalar. So, the resulting condition is (up to normalization)

$$\varphi_K^M \varphi_K^N - \phi_A^M \phi_A^N = \delta^{MN} . \quad (3.2)$$

Notice that this condition is invariant under *local* $SO(6)$ which allows to gauge away 15 additional scalars. Altogether 36 scalars are eliminated and the remaining $6n$ do indeed parametrize the coset $SO(6, n)/SO(6) \times SO(n)$.

Conditions (3.2) can be solved by first fixing an $SO(6)$ gauge such that the 6×6 matrix φ_N^M becomes symmetric, $\varphi = \varphi^T$, after which one can write down

$$\varphi = \sqrt{\mathbb{I} + \phi\phi^T} . \quad (3.3)$$

We can say that the $6n$ physical scalars ϕ are the *unconstrained* coordinates on the coset $SO(6, n)/SO(6) \times SO(n)$.

3.2 Harmonic description

The higher-derivative terms (2.19), (2.20) involve the function (potential) F defined on the coset of physical scalars. The peculiarity of this function is that it depends only on a single projection $Y_A^{++} = \phi_A^{++}(x, u) + \dots$ of the six-vectors of coset coordinates, obtained with the help of the $SU(4)$ harmonic variables. This is a typical example of an *analytic harmonic realization* of a coset space. Another, very similar example is that of the $\mathcal{N} = 4$ superconformal group $PSU(2, 2/4)$ realized on the Grassmann analytic superfields (2.9) (see Section 4). Here we explain this coset construction, following closely the case of $\mathcal{N} = 2$ superconformal symmetry and Poincaré supergravity [16, 17, 18] and of $\mathcal{N} = 2$ quaternionic sigma models [19, 20, 21].

We start by writing down the algebra of $SO(6, n)$ in a basis suitable for the forthcoming introduction of the harmonic variables (2.1). The $SO(n)$ generators are $M_{AB} = -M_{BA}$ and the $SO(6)$ ones are written in an $S(U(2) \times U(2))$ basis. Thus, the $S(U(2) \times U(2))$ generators are

$$Z_{+a}{}^{+b} \ (Z_{+a}{}^{+a} = 0), \quad Z_{-\dot{a}}{}^{-\dot{b}} \ (Z_{-\dot{a}}{}^{-\dot{a}} = 0), \quad Z_0 , \quad (3.4)$$

and the remaining generators of $SU(4)$ are $Z_{+a}{}^{-\dot{b}}$ and $Z_{-\dot{a}}{}^{+b}$. Finally, the generators of the coset $SO(6, n)/SO(6) \times SO(n)$ are $L_{Aa\dot{a}}$, L_{A++} and L_{A--} . Then the algebra of $SO(6, n)$ takes the form

$$\begin{aligned} [L_{Aa\dot{a}}, L_{Bb\dot{b}}] &= \delta_{AB} \left(\epsilon_{\dot{a}\dot{b}} \epsilon_{ac} Z_{+b}{}^{+c} + \epsilon_{ab} \epsilon_{\dot{a}\dot{c}} Z_{-\dot{b}}{}^{-\dot{c}} \right) + \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} M_{AB} \\ [L_{Aa\dot{a}}, L_{B++}] &= \delta_{AB} \epsilon_{\dot{a}\dot{b}} Z_{+a}{}^{-\dot{b}} \\ [L_{Aa\dot{a}}, L_{B--}] &= \delta_{AB} \epsilon_{ab} Z_{-\dot{a}}{}^{+b} \\ [L_{A++}, L_{B--}] &= \delta_{AB} Z_0 + M_{AB} \\ [Z_{+a}{}^{+b}, L_{Ac\dot{c}}] &= \delta_c^b L_{Aa\dot{c}} - \frac{1}{2} \delta_a^b L_{Ac\dot{c}} \\ [Z_{-\dot{a}}{}^{-\dot{b}}, L_{Ac\dot{c}}] &= \delta_c^{\dot{b}} L_{A\dot{a}c} - \frac{1}{2} \delta_{\dot{a}}^{\dot{b}} L_{Ac\dot{c}} \\ [Z_0, L_{A\pm\pm}] &= \pm 2 L_{A\pm\pm} \\ [Z_{+a}{}^{-\dot{b}}, L_{A--}] &= \epsilon^{\dot{b}c} L_{Aa\dot{c}} \end{aligned} \quad (3.5)$$

$$\begin{aligned}
[Z_{-\dot{a}}^{+b}, L_{A++}] &= \epsilon^{bc} L_{Ac\dot{a}} \\
[Z_{+a}^{-\dot{b}}, L_{Ac\dot{c}}] &= \epsilon_{ac} \delta_{\dot{a}}^{\dot{b}} L_{A++} \\
[Z_{-\dot{a}}^{+b}, L_{Ac\dot{c}}] &= \epsilon_{\dot{a}\dot{c}} \delta_c^b L_{A--} \\
[Z_{+a}^{+b}, Z_{+c}^{+d}] &= \delta_c^b Z_{+a}^{+d} - \delta_a^d Z_{+c}^{+b} \\
[Z_{-\dot{a}}^{-\dot{b}}, Z_{-\dot{c}}^{-\dot{d}}] &= \delta_{\dot{c}}^{\dot{b}} Z_{-\dot{a}}^{-\dot{d}} - \delta_{\dot{a}}^{\dot{d}} Z_{-\dot{c}}^{-\dot{b}} \\
[Z_0, Z_{+a}^{+b}] &= 2Z_{+a}^{+b} \\
[Z_0, Z_{-\dot{a}}^{+b}] &= -2Z_{-\dot{a}}^{+b}
\end{aligned}$$

Now, we want to realize this algebra on a coset of the group $SO(6, n)$. The standard coset $SO(6, n)/SO(6) \times SO(n)$ is obtained by putting all the generators M and Z in the coset denominator and leaving all the L 's in the coset with associated $6n$ coordinates ϕ :

$$\frac{SO(6, n)}{(M, Z)} \sim \{\phi_A^{++}, \phi_A^{--}, \phi_A^{a\dot{a}}\}. \quad (3.6)$$

We wish to have an alternative *S-analytic* coset involving only the coordinates ϕ_A^{++} associated with the generators L_{A++} . To this end we have to move the generators $L_{Aa\dot{a}}$, L_{A--} to the coset denominator. In doing this we encounter a problem: The $SO(6)$ generator $Z_{+a}^{-\dot{b}}$ converts $L_{Aa\dot{a}}$ into the coset generator L_{A++} . In order to avoid this, we proceed to the ‘harmonization’ of the coset. This means to introduce an additional group $\widehat{SU(4)}$ which we treat as independent of the $SO(6)$ from the coset denominator. Let us denote its generators by T_{+a}^{+b} , $T_{-\dot{a}}^{-\dot{b}}$, T_0 , $T_{+a}^{-\dot{b}}$ and $T_{-\dot{a}}^{+b}$, in complete analogy with $SO(6)$. We assume that this extra $\widehat{SU(4)}$ acts as an external automorphism of (3.5), i.e. $[T, Z] = Z$, $[T, L] = L$. Then it is clear that the combination $Z_{+a}^{-\dot{b}} - T_{+a}^{-\dot{b}}$ commutes with the generators of (3.5), in particular, with $L_{Aa\dot{a}}$. So, to avoid the above problem, we replace $Z_{+a}^{-\dot{b}}$ in the coset denominator by this combination. The group $\widehat{SU(4)}$ is itself realized on the harmonic coset $\widehat{SU(4)}/\widehat{S(U(2) \times U(2))}$, which means that we have to add the generators of the automorphism subgroup $\widehat{S(U(2) \times U(2))}$ to the coset denominator. The result is a particular *S-analytic* realization of the coset

$$\frac{SO(6, n) \otimes \widehat{SU(4)}}{(M, L_{Aa\dot{a}}, L_{A--}, Z_{+}^{+}, Z_{-}^{-}, Z_0, Z_{-}^{+}, Z_{+}^{-} - T_{+}^{-}, T_{+}^{+}, T_{-}^{-}, T_0)} \sim (\phi_A^{++}, w_i^{+a}, w_i^{-\dot{a}}) \quad (3.7)$$

parametrized by the coordinates ϕ_A^{++} associated with the $SO(6, n)$ generators L_{A++} and by harmonics $w_i^{+a}, w_i^{-\dot{a}}$ (the latter differ from the usual $SU(4)$ harmonics u (2.1), as explained below).

This coset is analytic in the sense that we consider functions $F(\phi_A^{++}, w)$ on it which are annihilated by the generators $L_{Aa\dot{a}}$ and L_{A--} . Then the algebra (3.5) implies

$$L_{Aa\dot{a}}F = L_{A--}F = 0 \Rightarrow M_{AB}F = Z_{+a}^{+b}F = Z_{-\dot{a}}^{-\dot{b}}F = Z_{-\dot{a}}^{+b}F = 0, \quad (3.8)$$

i.e., F cannot carry $SO(n) \times SU(2) \times SU(2)$ indices, but can have $U(1)$ charges under both Z_0 and T_0 . In addition, we impose the coset defining constraint

$$(Z_{+a}^{-b} - T_{+a}^{-b})F = 0. \quad (3.9)$$

It leads to a particular mixing of the coordinates associated with the $SO(6)$ generators Z and with the $\widehat{SU(4)}$ generators T . For this reason (3.7) is a semi-direct product (denoted by \ltimes in (3.7)) of the two cosets $SO(6, n)/SO(6) \times SO(n)$ and $\widehat{SU(4)}/\widehat{S(U(2) \times U(2))}$.

The actual construction of the coset goes through the following steps. We first introduce a *double harmonic space* involving, in addition to the $\widehat{SU(4)}$ harmonic variables u , harmonics κ_I^i on $SU(4) \sim SO(6)$ satisfying the defining conditions (cf. (2.1))

$$\kappa_I^i \bar{\kappa}_i^J = \delta_I^J, \quad \bar{\kappa}_i^I \kappa_I^J = \delta_i^j, \quad \epsilon^{IJKL} \kappa_I^i \kappa_J^j \kappa_K^k \kappa_L^l = \epsilon^{ijkl}. \quad (3.10)$$

They undergo $SU(4)$ transformations of two types: local (in the sense of $SU(4) \sim SO(6)$ from the coset denominator) with parameter λ and rigid with parameter σ :

$$\delta \kappa_I^i = \lambda_I^J \kappa_J^i + \kappa_I^j \sigma_j^i. \quad (3.11)$$

Our task now will be to make a change of variables from κ, u to z, w which are inert under the rigid $SU(4)$ and have simple transformation properties under the local $SU(4)$. This will allow us to impose the coset constraint (3.9) in a covariant way. We start by projecting the harmonics κ with u, \bar{u} :

$$\begin{aligned} \kappa_{+a}^{+b} &= \bar{u}_{+a}^I \kappa_I^i u_i^{+b}, & \kappa_{+a}^{-b} &= \bar{u}_{+a}^I \kappa_I^i u_i^{-b} \\ \kappa_{-\dot{a}}^{+b} &= \bar{u}_{-\dot{a}}^I \kappa_I^i u_i^{+b}, & \kappa_{-\dot{a}}^{-b} &= \bar{u}_{-\dot{a}}^I \kappa_I^i u_i^{-b} \end{aligned} \quad (3.12)$$

and similarly for the conjugate matrix $\bar{\kappa}$. Next we make the following non-linear change of variables (to simplify the notation, we suppress the $SU(2) \times SU(2)$ indices; the position of the $U(1)$ charges allows to unambiguously restore them):

$$\begin{aligned} z_-^+ &= \kappa_-^+(\kappa_+^+)^{-1} = -(\bar{\kappa}_-^-)^{-1} \bar{\kappa}_-^+, & z_-^- &= \bar{\kappa}_-^- \\ z_+^- &= \kappa_+^-(\kappa_-^-)^{-1} = -(\kappa_+^+)^{-1} \bar{\kappa}_+^-, & z_+^+ &= \bar{\kappa}_+^+. \end{aligned} \quad (3.13)$$

These new variables satisfy an algebraic constraint following from the fact that $\kappa \in SU(4)$, i.e. $\det \kappa = 1$. It can be used to eliminate, e.g. $\det z_-^-$ while the remaining $z_0 \equiv \det z_+^+$ can be treated as the coordinate of the $U(1)$ factor in $S(U(2) \times U(2)) \subset SU(4)$.

It is then not hard to check that the new variables z transform in the following way under the local $SU(4)$:

$$\begin{aligned} \delta z_-^+ &= \hat{\lambda}_-^+, & \delta z_-^- &= z_-^- \hat{\lambda}_-^-, & \delta z_+^+ &= \hat{\lambda}_+^+ z_+^+ \\ \delta z_+^- &= \hat{\lambda}_+^+ z_+^- + z_+^- \hat{\lambda}_-^- - \hat{\lambda}_+^-, \end{aligned} \quad (3.14)$$

where $\hat{\lambda}_\pm^\pm = \bar{w}_\pm^I \lambda_I^J w_J^\pm$ and we have introduced the *new harmonics*

$$\begin{aligned} w_i^{+a} &= u_i^{+a} + u_i^{-b} z_{-b}^{+a}, & w_i^{-\dot{a}} &= u_i^{-\dot{a}} \\ \bar{w}_{+a}^i &= \bar{u}_{+a}^i, & \bar{w}_{-\dot{a}}^i &= \bar{u}_{-\dot{a}}^i - z_{-\dot{a}}^{+b} \bar{u}_{+b}^i \end{aligned} \quad (3.15)$$

with transformation laws

$$\begin{aligned}\delta w_i^{+a} &= w_i^{-\dot{b}} \hat{\lambda}_{-\dot{b}}^{+a}, & \delta w_i^{-\dot{a}} &= 0 \\ \delta \bar{w}_{+a}^{\dot{i}} &= 0, & \delta \bar{w}_{-\dot{a}}^i &= -\hat{\lambda}_{-\dot{a}}^{+b} \bar{w}_{+b}^{\dot{i}}.\end{aligned}\tag{3.16}$$

We point out that these new harmonics are not unitary anymore (i.e., \bar{w} is not the conjugate of w), but they still satisfy the same algebraic relations as the unitary harmonics u (2.1).

What we have achieved is that the new variables do not mix under the local $SU(4)$ transformations with parameters $\hat{\lambda}$. This allows us to eliminate all of the z variables (with the exception of z_0) in a covariant way, which corresponds to imposing the Z coset conditions from (3.8) and the $Z - T$ condition (3.9).

3.3 Covariant constraints on the function F

Now we are able to see how the naive constraints (2.31), (2.34), (2.35) are modified due to the curvature of the coset space (3.7) on which the reduced function \mathcal{F} (2.29) lives. The origin of these constraints can be traced back to the S-analyticity conditions satisfied by the gauge-fixed function F (2.26). On the curved manifold they become covariant constraints (cf. (3.8)):

$$\mathcal{D}_{Aa\dot{a}} F = \mathcal{D}_{A--} F = 0.\tag{3.17}$$

Here \mathcal{D}_{AM} are covariant derivatives generalizing the flat derivatives $\partial/\partial\phi$. They satisfy the same $SO(6, n)$ algebra as the generators L_{AM} .

Let us start with the constraint (2.35). The second-order derivative in it can be rewritten as follows:

$$\begin{aligned}\mathcal{D}_{AM} \mathcal{D}_{BM} F &= (\mathcal{D}_{A++} \mathcal{D}_{B--} + \mathcal{D}_{A--} \mathcal{D}_{B++} - 2\mathcal{D}_{Aa\dot{a}} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \mathcal{D}_{B\dot{b}\dot{b}}) F \\ &= [\mathcal{D}_{A--}, \mathcal{D}_{B++}] F = -\delta_{AB} Z_0 F,\end{aligned}\tag{3.18}$$

where we have used the S-analyticity constraints (3.17) and the algebra (3.5). The function $F^{-4(g-1)}$ has two independent $U(1)$ charges, one with respect to the generator T_0 , $T_0 F^{-4(g-1)} = -4(g-1) F^{-4(g-1)}$ and the other for Z_0 . For a reason which will become clear in the next section, the Z_0 charge takes a different value, $Z_0 F = -4(g+1) F$. Thus, we have

$$\mathcal{D}_{AM} \mathcal{D}_{BM} F = 4(g+1) \delta_{AB} F,\tag{3.19}$$

or, in $SU(4)$ notation,

$$\epsilon^{pqrs} \mathcal{D}_{Apq} \mathcal{D}_{Br s} F = 32(g+1) \delta_{AB} F.\tag{3.20}$$

Further, the second-order derivative in (2.34) is replaced by $\mathcal{D}_{A[M} \mathcal{D}_{B N]} F$. Due to the constraints (3.17), this operator has only two non-vanishing projections obtained by taking $M = ++$ and $N = --$ or $N = a\dot{a}$. The first choice yields back the constraint (3.18), while the second gives rise to the commutator

$$[\mathcal{D}_{Aa\dot{a}}, \mathcal{D}_{B++}] F = \delta_{AB} \epsilon_{\dot{a}\dot{b}} \mathcal{Z}_{+a}^{-\dot{b}} F\tag{3.21}$$

where \mathcal{Z} is the covariant derivative replacing the generator Z . The effect of this is just a particular $SO(6)$ transformation of the coset coordinates, hence it is not really a constraint on the function.

Finally, in eq. (2.31) (or (2.32)) the flat partial derivative with respect to scalars is replaced by a covariant derivative

$$\epsilon^{pqrs} \frac{\partial}{\partial \bar{u}_{+a}^q} \mathcal{D}_{Ars} \mathcal{F} = 0. \quad (3.22)$$

We would like to point out that in the string theory analysis given in the following sections, the differential equations are obtained on functions \mathcal{F} which is the relevant part of F that survives the harmonic space integrals. Indeed string theory amplitudes directly see \mathcal{F} . The crucial step used in equation (3.18) was that F does not depend on 5 combinations of moduli as is expressed in the S-analyticity constraint (3.17). It is easy to see that \mathcal{F} does not satisfy this S-analyticity constraint since it is obtained by making a certain $SU(4)$ projection on F . Therefore the individual steps in this derivation cannot be applied to \mathcal{F} . However, the second order differential operators considered here are not sensitive to any particular $SU(4)$ projection of F and therefore the final equations are still true on \mathcal{F} .

4 $\mathcal{N} = 4$ conformal supersymmetry and supergravity

Here we show that the realization of G-analytic superfields of the type (2.9) as functions on a particular coset of the $\mathcal{N} = 4$ conformal superalgebra $PSU(2, 2/4)$ is very similar to the bosonic coset construction of the preceding section. This algebra involves the generators of Lorentz transformations ($M_{\mu\nu}$), translations (P_μ), conformal boosts (K^μ), dilatation (D), R symmetry $SU(4)$ (Z_i^j), Poincaré supersymmetry (Q_i^α and $\bar{Q}_{\dot{\alpha}}^i$) and special conformal supersymmetry (S_α^i and $\bar{S}_{\dot{\alpha}}^i$) with anticommutation relations for the odd generators (schematically)

$$\begin{aligned} \{Q, \bar{Q}\} &= P \\ \{S, \bar{S}\} &= K \\ \{Q, S\} &= M + D + Z. \end{aligned} \quad (4.1)$$

The standard superspace corresponds to the coset

$$\frac{PSU(2, 2/4)}{(M, K, D, S, \bar{S}, Z)} \sim (x^\mu, \theta_\alpha^i, \bar{\theta}_{\dot{\alpha}}^i) \quad (4.2)$$

involving all the 16 Grassmann variables associated with the supersymmetry generators. In order to obtain G-analytic superfields depending on half of these Grassmann variables, we add the $SU(4)$ harmonic projections of the Q generators $Q_{-\dot{a}}^\alpha = \bar{u}_{-\dot{a}}^i Q_i^\alpha$ and $\bar{Q}_{\dot{\alpha}}^{+a} = \bar{Q}_{\dot{\alpha}}^i u_i^{+a}$ to the coset denominator, thus leaving only the odd coordinates θ_α^{+a} and $\bar{\theta}_{-\dot{a}}^{\dot{\alpha}}$ in the coset. However, exactly as in the bosonic case of Section 3, the $SU(4)$ generator Z_+^- converts Q_- and \bar{Q}^+ from the coset denominator into the coset generators Q_+ and \bar{Q}^- . In order to

avoid this, we introduce the external automorphism group $\widehat{SU(4)}$ with generators T . Then the combination $Z_+^- - T_+^-$ commutes with all the Q 's and thus can be safely put in the coset denominator:⁵

$$\frac{PSU(2, 2/4) \otimes \widehat{SU(4)}}{(M, K, D, S, \bar{S}, Q_-, \bar{Q}^+, Z_+^+, Z_-^-, Z_0, Z_-^+, Z_+^- - T_+^-, T_+^+, T_-^-, T_0)} \sim (x, \theta^+, \bar{\theta}_-, w) . \quad (4.3)$$

Here the harmonics w are defined in exactly the same way as in Section 3, eq. (3.15), replacing the $SO(6)$ harmonics κ by R-symmetry $SU(4)$ harmonics. They transform as in (3.16) with the parameter $\hat{\lambda}$ replaced by the G-analytic superparameter

$$\Lambda_{-b}^{+a}(x, \theta^+, \bar{\theta}_-, w) = \bar{w}_{-b}^i \lambda_i^j w_j^{+a} + i\theta^{+a} \sigma^\mu \bar{\theta}_{-b} k_\mu + i\bar{w}_{-b}^i \eta_i^\alpha \theta_\alpha^{+a} + i\bar{\theta}_{-b}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}}^i w_i^{+a} \quad (4.4)$$

containing the parameters λ of the R-symmetry $SU(4)$, k of conformal boosts and η of special conformal supersymmetry.

The basic G-analytic conformal superfield $Y^{++}(x, \theta^+, \bar{\theta}_-, w)$ (2.9) (with superconformal harmonics w instead of u) describes the vector supermultiplet. It transforms with a G-analytic superconformal weight factor:

$$\begin{aligned} \delta Y^{++} &= Y^{++'}(x', \theta', \bar{\theta}', w') - Y^{++}(x, \theta, \bar{\theta}, w) = \Lambda Y^{++} \\ \Lambda(x, \theta^+, \bar{\theta}_-, w) &= \rho + k_\mu x^\mu + \bar{w}_{+a}^i \lambda_i^j w_j^{+a} + i\bar{w}_{+a}^i \eta_i^\alpha \theta_\alpha^{+a} + i\bar{\theta}_{-b}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}}^i w_i^{+a} \end{aligned} \quad (4.5)$$

where ρ is the parameter of dilatations.⁶ The other G-analytic object we are discussing here is the descendant $K_{\mu\nu}^{++}$ (2.14) of the Weyl multiplet. It is superconformal covariant due to the on-shell constraint and transforms with weight two, according to its scaling dimension, $\delta K^{++} = 2\Lambda K^{++}$.

The generalization to $\mathcal{N} = 4$ conformal supergravity is done by replacing the parameters Λ_{-b}^{+a} and Λ by arbitrary G-analytic superfields. Poincaré supergravity is obtained by coupling the Weyl multiplet to a set of six compensating vector multiplets (cf. (2.9))

$$y_{ij}^{++}(x, \theta^+, \bar{\theta}_-, w) = -y_{ji}^{++} = \varphi_{ij}^{kl} w_k^{+a} w_l^{+b} \epsilon_{ab} + \theta \text{ terms} . \quad (4.6)$$

Here we see the 6×6 matrix of compensating scalars φ_{ij}^{kl} . Let us consider the following projections of y_{ij}^{++} with the harmonics w :

$$y_{a\dot{a}}^{++} = \bar{w}_{+a}^i \bar{w}_{-\dot{a}}^j y_{ij}^{++} , \quad y_0 = \frac{1}{2} \epsilon^{ab} \bar{w}_{+a}^i \bar{w}_{+b}^j y_{ij}^{++} . \quad (4.7)$$

It is easy to check that they transform as follows:

$$\delta y_{a\dot{a}}^{++} = \epsilon_{ab} \Lambda_{-\dot{a}}^{+b} y_0 + \Lambda y_{a\dot{a}}^{++} , \quad \delta y_0 = \Lambda y_0 , \quad (4.8)$$

⁵Here we follow the formulation of $\mathcal{N} = 2$ conformal supersymmetry of [16, 18]. A somewhat different approach is proposed in [9].

⁶It can be shown that $\Lambda_{-b}^{+a} = D_{-b}^{+a} \Lambda$.

so their ratio transforms as a *compensator* for the local superconformal transformations:

$$\delta \left(\frac{y_{a\dot{a}}^{++}}{y_0} \right) = \epsilon_{ab} \Lambda_{-\dot{a}}^{+b} . \quad (4.9)$$

Then, with the help of this compensator we can define new harmonics *inert under the local superconformal transformations* (notice the similarity with (3.15) and (3.16)):

$$\begin{aligned} v_i^{+a} &= w_i^{+a} - w_i^{-\dot{b}} \epsilon^{ab} \frac{y_{b\dot{b}}^{++}}{y_0}, \quad v_i^{-\dot{a}} = w_i^{-\dot{a}} \\ \bar{v}_{+a}^{\dot{i}} &= \bar{w}_{+a}^{\dot{i}}, \quad \bar{v}_{-\dot{a}}^i = \bar{w}_{-\dot{a}}^i + \epsilon^{ab} \frac{y_{b\dot{a}}^{++}}{y_0} \bar{w}_{+a}^i \\ \delta v &= \delta \bar{v} = 0 . \end{aligned} \quad (4.10)$$

The role of the compensators is to completely absorb the local superconformal transformations. This allows us to use the parameter $\Lambda_{-\dot{a}}^{+b}$ in (4.9) fix a gauge in which $y_{a\dot{a}}^{++} = 0$, thus identifying the harmonics v and w . This means, in particular, that the conformal $SU(4)$ (generators Z in (4.3)) is identified with $\widehat{SU(4)}$ (generators T in (4.3)). By the same logic, we can use the parameter $\hat{\Lambda}_-^{+}$ of local $SO(6)$ transformations in (3.14) to gauge away the compensator z_-^{+} . This results in the identification of the harmonics w with u . So, at the expense of manifest covariance, the different $SU(4)$ groups discussed above are reduced to a unique one, and the harmonics to the original ones (2.1). This gauge fixing procedure establishes a bridge between the S-analytic coset (3.7) and the G-analytic coset (4.3).

Finally, we are ready for the superconformal covariantization of the higher-derivative terms (2.19), (2.20). It is achieved in three steps. Firstly, we replace the explicit harmonics u in $F(Y, u)$ by the new inert ones v (however, the superfields Y still depend on the conformal harmonics w). Secondly, we introduce weightless G-analytic superfields Y/y_0 . In this way the potential $F(Y, v)$ becomes conformal invariant. Thirdly, we use the G-analytic density y_0 to compensate the weight $4(g+1)$ of the Weyl factor (the measure is weightless, as can be seen from its vanishing scaling dimension). The result is

$$\begin{aligned} S_1 &= \int d^4x du d^4\theta^+ d^4\bar{\theta}_- (\bar{K}_{\mu\nu}^{++} \bar{K}^{++\mu\nu}) (y_0)^{-4(g+1)} (K_{\rho\sigma}^{++} K^{++\rho\sigma})^g F_1^{-4(g-1)} \left(\frac{Y_A^{++}}{y_0}, v \right), \\ S_2 &= \int d^4x du d^4\theta^+ d^4\bar{\theta}_- (K_{\mu\nu}^{++} K^{++\mu\nu})^{g+1} (y_0)^{-4(g+1)} F_2^{-4(g-1)} \left(\frac{Y_A^{++}}{y_0}, v \right). \end{aligned} \quad (4.11)$$

The presence of the density $(y_0)^{-4(g+1)}$ in (4.11) explains why in (3.19) we took the value $Z_0 F = -4(g+1)F$ of the charge Z_0 , different from that of the charge T_0 . This density should be viewed as part of the covariantized function F discussed at the end of Section 3. Then, F is a function of the G-analytic superfields Y_A^{++} and y_{ij}^{++} and hence is a G-analytic superconformal object itself. This means that it is annihilated by the supercharges Q_-, \bar{Q}^+ from the coset denominator in (4.3). This is compatible with the

condition of superconformal primarity (that the object is annihilated by all the special superconformal charges S) only if the dilatation and Z_0 weights of the object coincide [18, 22]. Finally, the local $SU(4)$ gauge-fixing procedure (elimination of the compensators) results in the identification of the Z_0 charges from (3.7) and (4.3). The automorphism charge T_0 remains independent and, indeed, takes a different value.⁷

5 Topological amplitudes - review

In Sections 2 and 3 it was argued from the general structure (2.29) of the harmonic expansion of the supergravity amplitudes $\mathcal{F}_g^{(1,3)}$ that they fulfill differential equations of first order (2.31) and second order (3.20) in the moduli of the internal compactification manifold (i.e. $K3 \times T^2$ for type II string theory). In this section, we would like to check these relations by applying them directly to the string amplitudes. Since, as we have already pointed out, the latter are captured by correlation functions of the topological string, it would be logical, to consider the twisted version of the theory. However, here we are facing the problem that some of the moduli involved in the $K3 \times T^2$ compactification are in fact part of the Ramond-Ramond sector of the theory, for which we have at present no representation in terms of the $\mathcal{N} = 4$ superconformal algebra, which is used to formulate the topological correlators. Besides that, the direct study of (2.31) and (3.20) in the untwisted version of the type II string is quite cumbersome, since we would have to deal with (in principle) an arbitrary high number of loops.

Fortunately, as was found in [5], the dual amplitudes of the couplings (2.20) in the heterotic theory compactified on T^6 begin receiving corrections already at the 1-loop level, which are relatively simple to compute. Therefore, for the purpose of checking (2.31) and (3.20), we will focus on this amplitude which we review below.

After performing explicitly the superspace integrals of the 1/2-BPS F -type term (2.20) we encounter among many different contributions a coupling of two self-dual Riemann tensors, two graviscalars and $2g - 2$ graviphoton field strengths at $(g + 1)$ -loop order

$$S_2 = \int d^4x \mathcal{F}_g^{(3)} R_{(+)}^2 (\partial\partial\Phi)^2 (T_{(+)}^{++})^{2g-2}, \quad (5.1)$$

where we remind that $\mathcal{F}_g^{(3)}$ corresponds in the supergravity context to the reduced part of $F_2^{-4(g-1)}$. The corresponding heterotic string 1-loop torus amplitude can be formulated as the following two-dimensional integral over the fundamental domain \mathbb{F} of the world-sheet torus

$$\mathcal{F}_g^{(\text{HET})} = \int_{\mathbb{F}} \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R) \in \Gamma(6, 22)} \left(\frac{1}{2} \bar{u}_{++}^{ij} P_{ij}^L \right)^{2g-2} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2}. \quad (5.2)$$

⁷This situation is different from $\mathcal{N} = 2$ superconformal symmetry where the relevant G-analytic superfields, e.g. the hypermultiplet, have equal Z_0 and T_0 charges [18]. This can be explained by the different properties of the G-analytic superspace measures – the $\mathcal{N} = 2$ measure has a conformal weight while the $\mathcal{N} = 4$ one does not.

In this expression $\tau = \tau_1 + i\tau_2$ is the Teichmüller parameter of the torus, while $q = e^{2\pi i\tau}$. Moreover, $\eta(\tau)$ is the Dedekind eta-function given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad (5.3)$$

and G_{g+1} is defined via the following expansion of a generating functional for space-time correlation functions

$$\begin{aligned} G(\lambda, \tau, \bar{\tau}) &\equiv \sum_{h=0}^{\infty} \frac{1}{(h!)^2} \left(\frac{\lambda}{\tau_2} \right)^{2h} \left\langle \prod_{i=1}^h \int d^2 x_i X^1 \bar{\partial} X^2(x_i) \prod_{j=1}^h \int d^2 y_j \bar{X}^2 \partial \bar{X}^1(y_j) \right\rangle = \\ &= \sum_{h=1}^{\infty} \lambda^{2h} G_h(\tau, \bar{\tau}). \end{aligned} \quad (5.4)$$

In [23], this generating functional was calculated with the result

$$G(\lambda, \tau, \bar{\tau}) = \left(\frac{2\pi i \lambda \bar{\eta}^3}{\vartheta(\lambda, \bar{\tau})} \right)^2 \exp \left(-\frac{\pi \lambda^2}{\tau_2} \right), \quad (5.5)$$

where ϑ is the usual odd theta-function defined by

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n - \frac{1}{2})^2} e^{2\pi i(z - \frac{1}{2})(n - \frac{1}{2})}. \quad (5.6)$$

The most important property of G_g for our purposes is the fact that upon differentiation with respect to τ it becomes G_{g-1} :

$$\frac{\partial}{\partial \tau} G_g = -\frac{i\pi}{2\tau_2^2} G_{g-1}. \quad (5.7)$$

In (5.2), \bar{u}_{++}^{ij} are precisely the harmonics of the coset $\frac{SU(4)}{S(U(2) \times U(2))}$, which appear in the reduced harmonic expansion of the $\mathcal{F}_g^{(\text{HET})}$ in (2.29). Finally, P_{ij}^L and P_A^R are the left- and right-moving momenta of a $\Gamma^{(6,22)}$ Narain-lattice describing the compactification of the heterotic string on the T^6 torus. They encode the full dependence of the amplitude on the corresponding $6 \times 22 = 132$ moduli, which form the manifold

$$\mathcal{M} = \frac{SO(6, 22)}{SO(6) \times SO(22)}, \quad (5.8)$$

as explained in Section 3.1. The exact parameterization of the lattice momenta, however, will be of no importance to our calculations and would involve the explicit construction of the world-sheet sigma model action, starting from the four-dimensional action of $\mathcal{N} = 4$

supergravity coupled to 22 vector multiplets. The left-moving momenta P_{ij}^L are formulated in a complex $SU(4)$ basis and their square is given by

$$(P^L)^2 = \frac{1}{8} \epsilon^{ijkl} P_{ij}^L P_{kl}^L, \quad (5.9)$$

which is manifestly real and $SU(4)$ invariant. Moreover, in order to streamline our notation, we will also introduce the following projection of the momenta

$$P_{++}^L \equiv \frac{1}{2} \bar{u}_{++}^{ij} P_{ij}^L. \quad (5.10)$$

6 First-order harmonicity relation

With the above setting, we are now in a position to discuss the harmonicity equation (2.31) (or (3.22)). In [5], it was shown that $\mathcal{F}_g^{(\text{HET})}$ satisfy the following relation

$$\epsilon^{ijkl} \frac{\partial}{\partial \bar{u}_{+1}^i} \frac{\partial}{\partial \bar{u}_{+2}^j} D_{kl,A} \mathcal{F}_g^{(\text{HET})} = 0, \quad (6.1)$$

up to an anomaly, which was calculated explicitly. The action of the differential $D_{ij,A}$ with respect to the moduli $\phi_{ij,A}$ can be analyzed in two different ways:

- From the world-sheet point of view, it amounts inserting the scalar vertex operator

$$V_{ij,A}^{\text{mod.}} = -\frac{1}{2\pi} \partial X_{ij} \bar{J}_A(z) e^{ip \cdot X}, \quad (6.2)$$

into the correlation function, where X_{ij} are the internal bosonic coordinates in an $SU(4)$ basis, satisfying the pseudo-reality condition

$$\overline{X^{ij}} = \frac{1}{2} \epsilon^{ijkl} X_{kl}, \quad (6.3)$$

and \bar{J}_A are the right-moving (Abelian) currents.

This approach is rather cumbersome, since the correlator corresponding to e.g. (6.1) contains $(2g+3)$ vertices, for which all possible contractions need to be considered. We will therefore rather resort to the following approach.

- In terms of the $\Gamma^{(6,22)}$ lattice momenta, the differentials act as infinitesimal Lorentz boosts⁸

$$D_{ij,A} P_{kl}^L = \epsilon_{ijkl} P_A^R, \quad D_{ij,A} P_B^R = \frac{\delta_{AB}}{2} P_{ij}^L. \quad (6.4)$$

⁸Note a factor of 2 misprint in eq. (10.7) of [5] which had no effect in the subsequent analysis.

These rules were proved in [5] by an explicit world-sheet computation at the linearized level. It can be easily checked that they in fact reproduce the algebra (3.5), up to normalization factors. Moreover, they annihilate the $SO(6, 22)$ -square of the lattice vectors

$$D_{ij,A} \left((P^L)^2 - (P^R)^2 \right) = 0. \quad (6.5)$$

As we have seen in Section 2, the general harmonic expansion of $\mathcal{F}_g^{(\text{HET})}$ suggests that (6.1) is in fact merely a consequence of the stronger relation (2.31). The goal of this Section is to explicitly test the validity of (2.31) and to examine whether its right hand side is modified by an anomaly as it was the case for (6.1).

The computation is done in a straight-forward way using the differentiation rules (6.4)

$$\begin{aligned} E_1 &\equiv \epsilon_{ab} \epsilon^{ijkl} \frac{\partial}{\partial \bar{u}_{+b}^j} D_{kl,A} \mathcal{F}_g^{(\text{HET})} = \\ &= \epsilon_{ab} \epsilon^{ijkl} \frac{\partial}{\partial \bar{u}_{+b}^j} D_{kl,A} \int \frac{d^2 \tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} (P_{++}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\ &= 2\epsilon_{ab} \frac{\partial}{\partial \bar{u}_{+b}^j} \int \frac{d^2 \tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} (P_{++}^L)^{2g-3} \left[(2g-2) \bar{u}_{++}^{ij} - \pi \tau_2 \epsilon^{ijkl} P_{kl}^L (P_{++}^L) \right] \cdot \\ &\quad \cdot P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}, \end{aligned} \quad (6.6)$$

which was essentially already found in [5]. Using the simple identity

$$\epsilon_{ab} \frac{\partial}{\partial \bar{u}_{+b}^j} P_{++}^L = \bar{u}_{+a}^i P_{ij}^L, \quad (6.7)$$

we can easily calculate the harmonic partial derivative

$$\begin{aligned} E_1 &= 2 \int \frac{d^2 \tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left[3(2g-2) \bar{u}_{+a}^i (P_{++}^L) - (2g-2) \pi \tau_2 \bar{u}_{+a}^m P_{mj}^L \epsilon^{ijkl} P_{kl}^L (P_{++}^L) + \right. \\ &\quad \left. + (2g-3)(2g-2) \bar{u}_{+a}^m P_{mj}^L \bar{u}_{++}^{ij} \right] (P_{++}^L)^{2g-4} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \end{aligned}$$

Using furthermore the trivial relation

$$\epsilon^{inkl} P_{jn}^L P_{kl}^L = 2\delta_j^i (P^L)^2, \quad (6.8)$$

we can further simplify the expression

$$\begin{aligned} E_1 &= 2(2g-2) \bar{u}_{+a}^i \int \frac{d^2 \tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \left[2g \tau_2^{2g-1} - 2\pi \tau_2^{2g} (P^L)^2 \right] (P_{++}^L)^{2g-3} \cdot \\ &\quad \cdot P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\ &= 4i(2g-2) \bar{u}_{+a}^i \int \frac{d^2 \tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \frac{\partial}{\partial \tau} \left[\tau^{2g} (P_{++}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right]. \end{aligned}$$

At this point, we perform a partial integration in τ and use modular invariance together with the exponential suppression in the infra-red region $\tau_2 \rightarrow \infty$, due to the presence of P_L for $g > 1$, to conclude that there are no boundary terms we have to worry about.⁹ The only contribution therefore comes when the τ -derivative acts on G_{g+1} . Using the identity (5.7) we get

$$E_1 = -2(2g-2)\pi\bar{u}_{+a}^i \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \quad (6.9)$$

This result has to be contrasted with the expression

$$D_{++,A} \mathcal{F}_{g-1}^{(\text{HET})}, \quad (6.10)$$

where we have used the same projection as in (5.10)

$$D_{++,A} \equiv \frac{1}{2} \bar{u}_{++}^{ij} D_{ij,A}. \quad (6.11)$$

The calculation follows much along the same lines as before and yields the result

$$D_{++,A} \mathcal{F}_{g-1}^{(\text{HET})} = -2\pi \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \quad (6.12)$$

Comparing this result with (6.9), one concludes

$$\epsilon_{ab} \epsilon^{ijkl} \frac{\partial}{\partial \bar{u}_{+b}^j} D_{kl,A} \mathcal{F}_g^{(\text{HET})} = (2g-2) \bar{u}_{+a}^i D_{++,A} \mathcal{F}_{g-1}^{(\text{HET})}. \quad (6.13)$$

Since the $\mathcal{F}_{g-1}^{(\text{HET})}$, which appears on the right hand side is of lower order in g than the initial one we considered on the left hand side, this term can be interpreted as an anomaly to the harmonicity relation. This is justified by comparison to the holomorphic anomaly equation [1]-[4], where (for the type II theory) the lower genus¹⁰ terms have their origin from boundary contributions in the moduli space of genus g world-sheets.

As a trivial consistency check of this result, we can try to recover the weaker harmonicity relation presented in [5], by applying a second partial differentiation with respect to \bar{u}_{+a}^i to (6.13) using the fact that it commutes with $\frac{\partial}{\partial \bar{u}_{+b}^j} D_{kl,A}$

$$\epsilon_{ab} \epsilon^{ijkl} \frac{\partial}{\partial \bar{u}_{+a}^i} \frac{\partial}{\partial \bar{u}_{+b}^j} D_{kl,A} \mathcal{F}_g^{(\text{HET})} = 2(2g-2)(2g+1) D_{++,A} \mathcal{F}_{g-1}^{(\text{HET})}, \quad (6.14)$$

which is precisely the result found in [5].

⁹Note that the equation is trivially fulfilled in the case $g = 1$, since $\mathcal{F}_1^{(\text{HET})}$ is independent of the harmonic variables.

¹⁰In [2] scattering amplitudes of two (self-dual) Riemann tensors and $(2g-2)$ graviphoton field strengths in type II theory compactified on Calabi-Yau threefolds were considered. In these amplitudes, the number g corresponds to the genus of the world-sheet Riemann surface.

7 Second-order constraint

In the same way as equation (2.31), we can now check relation (2.33) (or rather its counterparts (3.20) and (3.21) taking into account the curvature of the moduli space) by directly applying the corresponding differential operator to the topological amplitude $\mathcal{F}_g^{(\text{HET})}$. We use again the differentiation rules (6.4) to obtain

$$\begin{aligned} E_2 &\equiv \epsilon^{ijkm} D_{ij,A} D_{kl,B} \mathcal{F}_g^{(\text{HET})} = \\ &= \epsilon^{ijkm} D_{ij,A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left[\frac{1}{2} (2g-2) \epsilon_{klpq} \bar{u}_{++}^{pq} - 2\pi\tau_2 P_{kl}^L (P_{++}^L) \right] (P_{++}^L)^{2g-3} \\ &\quad \cdot P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \end{aligned}$$

Taking now the second scalar derivative, one has

$$\begin{aligned} E_2 &= \epsilon^{ijkm} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left\{ \left[-2\pi\tau_2 \epsilon_{ijkl} (P_{++}^L) - \pi\tau_2 P_{kl}^L \epsilon_{ijpq} \bar{u}_{++}^{pq} \right] (P_{++}^L)^{2g-3} + \right. \\ &\quad \left. + \left[\frac{1}{2} (2g-2) \epsilon_{klpq} \bar{u}_{++}^{pq} - 2\pi\tau_2 P_{kl}^L (P_{++}^L) \right] \cdot \left[\frac{1}{2} (2g-3) \epsilon_{ijrs} \bar{u}_{++}^{rs} - 2\pi\tau_2 P_{ij}^L (P_{++}^L) \right] \right. \\ &\quad \left. \cdot (P_{++}^L)^{2g-4} \right\} \cdot P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} + \\ &\quad + \frac{\delta_{AB}}{2} \epsilon^{ijkm} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} P_{ij}^L \left[\frac{1}{2} (2g-2) \epsilon_{klpq} \bar{u}_{++}^{pq} - 2\pi\tau_2 P_{kl}^L (P_{++}^L) \right] \\ &\quad \cdot (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \end{aligned}$$

Regrouping the terms furthermore

$$\begin{aligned} E_2 &= \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left\{ -12\pi\tau_2 \delta_l^m (P_{++}^L)^{2g-2} - 4\pi(2g-2) \tau_2 P_{kl}^L \bar{u}_{++}^{km} (P_{++}^L)^{2g-3} - \right. \\ &\quad \left. - \pi\tau_2 (2g-2) \epsilon_{klpq} \epsilon^{ijkm} \bar{u}_{++}^{pq} P_{ij}^L (P_{++}^L)^{2g-3} + 4\pi^2 \tau_2^2 \epsilon^{ijkm} P_{kl}^L P_{ij}^L (P_{++}^L)^{2g-2} \right\} \\ &\quad \cdot P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} + \\ &\quad + \frac{\delta_{AB}}{2} [(2g-2) \epsilon^{ijkm} \epsilon_{klpq} P_{ij}^L \bar{u}_{++}^{pq} - 2\pi\tau_2 \epsilon^{ijkm} P_{ij}^L P_{kl}^L (P_{++}^L)] (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \end{aligned}$$

and using the relations

$$\epsilon^{klmi} P_{kl}^L P_{mj}^L = 2\delta_j^i (P^L)^2, \quad (7.1)$$

$$\frac{1}{2} \epsilon^{klmi} \epsilon_{mj pq} P_{kl}^L \bar{u}_{++}^{pq} + 2P_{pj}^L \bar{u}_{++}^{pi} = 2(P_{++}^L) \delta_j^i, \quad (7.2)$$

we obtain

$$\begin{aligned}
E_2 = & -4\pi \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g} G_{g+1} \sum_{(P^L, P^R)} [(2g+1) - 2\pi\tau_2(P^L)^2] \delta_l^m (P_{++}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} + \\
& + \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} [2g - 2\pi\tau_2(P^L)^2] \delta_m^l (P_{++}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
& - \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left[2(P_{++}^L) \delta_m^l + (2g-2) P_{pm}^L u_{++}^{pl} \right] (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}
\end{aligned} \tag{7.3}$$

At this point one can check that the first two lines as well as the last line, separately, are indeed modular invariant (for the last line, this follows mainly from the presence of the harmonics). Moreover, the first two lines can be written as differentiations with respect to the torus Teichmüller parameter:

$$\begin{aligned}
E_2 = & -8i\pi \delta_l^m \int \frac{d^2\tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \frac{\partial}{\partial\tau} \left[\tau_2^{2g+1} (P_{++}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right] + \\
& + 2i\delta_{AB} \delta_l^m \int \frac{d^2\tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \frac{\partial}{\partial\tau} \left[\tau_2^{2g} (P_{++}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right] - \\
& - \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left[2(P_{++}^L) \delta_m^l + (2g-2) P_{pm}^L u_{++}^{pl} \right] (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}.
\end{aligned} \tag{7.4}$$

Since these terms are modular invariant, one is allowed to perform a partial integration, with $\frac{\partial}{\partial\tau}$ only hitting the factor G_{g+1}

$$\begin{aligned}
E_2 = & 4\pi^2 \delta_l^m \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
& - \delta_{AB} \pi \delta_l^m \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
& - \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} \left[2(P_{++}^L) \delta_m^l + (2g-2) P_{pm}^L u_{++}^{pl} \right] (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}.
\end{aligned}$$

This expression can be contrasted with

$$D_{++,A} D_{++,B} \mathcal{F}_{g-1}^{(\text{HET})}, \tag{7.5}$$

which can be computed using exactly the same rules as before

$$\begin{aligned}
D_{++,A}D_{++,B}\mathcal{F}_{g-1}^{(\text{HET})} &= \\
&= -2\pi D_{++,A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-3} P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\
&= 4\pi^2 \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
&\quad - \pi \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{++}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}.
\end{aligned}$$

From this, we conclude for the second order constraint

$$\begin{aligned}
\epsilon^{ijkm} D_{ij,A} D_{kl,B} \mathcal{F}_g^{(\text{HET})} &= \delta_l^m D_{++,A} D_{++,B} \mathcal{F}_{g-1}^{(\text{HET})} - 2\delta_l^m \delta_{AB} \mathcal{F}_g^{(\text{HET})} - \\
&\quad - (2g-2) \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-1} G_{g+1} \sum_{(P^L, P^R)} P_{pm}^L \bar{u}_{++}^{pl} (P_{++}^L)^{2g-3} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \quad (7.6)
\end{aligned}$$

Notice that the last two terms are not of the form of an anomaly but are generic “hard” contributions to the equation. As one can easily see, they correspond however to an $SU(4)$ ($\sim SO(6)$) rotation acting on the harmonics inside $\mathcal{F}_g^{(\text{HET})}$, which is exactly what one expects according to (3.21).

On the other hand, (7.6) is vastly simplified when we contract its free $SU(4)$ indices:

$$\epsilon^{ijkl} D_{ij,A} D_{kl,B} \mathcal{F}_g^{(\text{HET})} = 4D_{++,A} D_{++,B} \mathcal{F}_{g-1}^{(\text{HET})} - 4(g+1) \delta_{AB} \mathcal{F}_g^{(\text{HET})}. \quad (7.7)$$

Comparing this result to (3.20), we conclude that besides the anomalous term proportional to $\mathcal{F}_{g-1}^{(\text{HET})}$ the two relations indeed agree, up to an irrelevant normalization.

Finally, let us mention in passing that a second order differentiation, which is antisymmetrized in the $SO(22)$ indices, is exactly vanishing

$$\frac{\partial}{\partial \phi_{[A}^{ij}} \frac{\partial}{\partial \phi_{B]}^{kl}} \mathcal{F}_g^{(\text{HET})} = 0. \quad (7.8)$$

This can be seen most easily by representing the above expression as a correlator with two additional scalar vertices inserted and realizing that its right-moving part is given by

$$\langle \bar{J}_{[A}(\bar{z}) \bar{J}_{B]}(\bar{w}) \rangle, \quad (7.9)$$

which follows from the form of the scalar vertex operator (6.2). Since the right-moving currents are Abelian, it follows that expression (7.9) is identically zero. Note in particular that in this case there is not even an anomaly, and (7.8) remains in fact exact at the quantum level.

8 Harmonicity in six dimensions

8.1 The origin of the harmonicity constraint

In this subsection we summarize a few key points about six-dimensional harmonic superspace and derive the corresponding harmonicity constraint. The discussion closely follows that of the $\mathcal{N} = 4$ case in four dimensions, therefore it is very brief.

We consider $\mathcal{N} = (1, 1)$ supersymmetry in six dimensions whose automorphism group is $SU(2)_L \times SU(2)_R$. Let us introduce harmonic variables v_a^I for $SU(2)_L$ and $v_{\dot{a}}^{\dot{I}}$ for $SU(2)_R$, together with their conjugates $v_I^a = (v_a^I)^*$ and $v_{\dot{I}}^{\dot{a}} = (v_{\dot{a}}^{\dot{I}})^*$. Here a, \dot{a} are $SU(2)$ doublet indices while $I, \dot{I} = 1, 2$ are projections onto the subgroup $U(1)_L \times U(1)_R$. They satisfy the completeness conditions

$$v_a^I v_J^a = \delta_J^I, \quad v_I^a v_b^I = \delta_b^a \quad (8.1)$$

(and similarly for $v_{\dot{a}}^{\dot{I}}$). Raising and lowering the indices with ϵ_{ab} , ϵ_{IJ} , etc., we can rewrite the non-trivial part of (8.1) as the unit determinant condition

$$\epsilon^{ab} v_a^1 v_b^2 = 1. \quad (8.2)$$

In fact, the harmonics can be viewed as matrices of the corresponding $SU(2)$ groups.

The harmonic functions are supposed to have harmonic expansions homogeneous under the action of the subgroup $U(1)_L \times U(1)_R$. For example, a function of unit $U(1)_L \times U(1)_R$ charges has the expansion

$$\phi^{1\dot{1}}(v) = \phi^{a\dot{a}} v_a^1 v_{\dot{a}}^{\dot{1}} + \phi^{(abc)\dot{a}} v_a^1 v_b^1 v_c^2 v_{\dot{a}}^{\dot{1}} + \phi^{a(\dot{a}\dot{b}\dot{c})} v_a^1 v_{\dot{a}}^{\dot{1}} v_{\dot{b}}^{\dot{1}} v_{\dot{c}}^{\dot{2}} + \dots, \quad (8.3)$$

so that in each term the number of v^1 exceeds by one the number of v^2 (the same for $v^{\dot{1}, \dot{2}}$). Notice that due to the constraint (8.2) each component is an irrep of $SU(2)_L \times SU(2)_R$ (i.e., only symmetrized indices appear). Effectively, such homogeneous functions live on the coset $S_L^2 \times S_R^2 = (SU(2)_L/U(1)_L) \times (SU(2)_R/U(1)_R)$.

The introduction of harmonic variables allows us to define G-analytic superfields which depend only on half of the Grassmann variables,¹¹ e.g. on $\theta_\alpha^1 = v_a^1 \theta_\alpha^a$, $\bar{\theta}_2^\alpha = v_{\dot{2}}^{\dot{\alpha}} \bar{\theta}_2^{\dot{\alpha}} = \bar{\theta}^{\alpha\dot{1}}$. One such short superfield describes the (on-shell) vector multiplet

$$Y^{1\dot{1}}(\theta^1, \bar{\theta}_2, v) = \phi^{a\dot{a}} v_a^1 v_{\dot{a}}^{\dot{1}} + \theta_\alpha^1 \bar{\psi}_{\dot{a}}^\alpha v_{\dot{2}}^{\dot{a}} + \bar{\theta}_2^\alpha \psi_\alpha^a v_a^1 + \bar{\theta}_2^\alpha \sigma^{\mu\nu} \theta^1 F_{\mu\nu} + \dots. \quad (8.4)$$

Notice the conservation of the overall charges $1, \dot{1}$ carried by the projected Grassmann variables or by the explicit harmonics projecting the component fields. This superfield is real in the sense $\widetilde{Y^{1\dot{1}}} = Y^{1\dot{1}}$, where \sim is a combination of complex conjugation with a reflection on $S^2 \times S^2$ preserving G-analyticity. In particular, this implies the reality of the first component, $(\phi^{a\dot{a}})^* = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} \phi^{\dot{b}b}$.

Another short superfield of the same type describes the (on-shell) Weyl multiplet [6]

$$(W^{1\dot{1}})_\alpha{}^\beta(\theta^1, \bar{\theta}_2, v) = (T^{a\dot{a}})_\alpha{}^\beta v_a^1 v_{\dot{a}}^{\dot{1}} + \theta_\gamma^1 \bar{\theta}_2^\delta R_{\alpha\delta}{}^{\beta\gamma} + \dots, \quad (8.5)$$

¹¹We use $SU^*(4) \sim SO(1, 5)$ chiral spinor notation with left-handed ψ_α and right-handed $\bar{\psi}^\alpha$ spinors.

where $(W^{1\dot{1}})_\alpha{}^\beta$ is in the adjoint of $SU^*(4)$ ($(W^{1\dot{1}})_\alpha{}^\alpha = 0$), $(T^{a\dot{a}})_\alpha{}^\beta = (T^{a\dot{a}})_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta$ are the graviphoton field strengths and $R_{\alpha\delta}{}^{\beta\gamma} = R_{\mu\nu\lambda\rho}(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma^{\lambda\rho})_\delta{}^\gamma$ is the curvature.

In [6] the following term of the six-dimensional effective action was considered:

$$\int d^6x \, dv \, d^4\theta^1 \, d^4\bar{\theta}_2 \left[\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon_{\beta_1\beta_2\beta_3\beta_4} (W^{1\dot{1}})_{\alpha_1}{}^{\beta_1} (W^{1\dot{1}})_{\alpha_2}{}^{\beta_2} (W^{1\dot{1}})_{\alpha_3}{}^{\beta_3} (W^{1\dot{1}})_{\alpha_4}{}^{\beta_4} \right]^g \times F_{(1)^{4g-4}(\dot{1})^{4g-4}}(Y^{1\dot{1}}, v) . \quad (8.6)$$

In fact, what appears in (8.6) is the determinant of the 4×4 traceless matrix $(W^{1\dot{1}})_\alpha{}^\beta$. This is a Lorentz invariant which breaks up into two independent invariants, $[\text{Tr}(W^{1\dot{1}})]^2$ and $\text{Tr}(W^{1\dot{1}})^4$. We could use anyone of them to construct an effective action term similar to (8.6). However, upon decompactification of the four-dimensional couplings (2.19) and (2.20), one can show that only the first of the two invariants contributes. We will eventually study this case in the next subsection. The corresponding effective action term is

$$\int d^6x \, dv \, d^4\theta^1 \, d^4\bar{\theta}_2 \left[(W^{1\dot{1}})_\alpha{}^\beta (W^{1\dot{1}})_\beta{}^\alpha \right]^{g+1} F_{(1)^{2g-2}(\dot{1})^{2g-2}}(Y^{1\dot{1}}, v) . \quad (8.7)$$

The function $F_{(1)^m(\dot{1})^m}$ ($m = 2g - 2$) has to carry a ‘negative’ (i.e. indices 1, $\dot{1}$ downstairs) charges of each kind, in order to compensate that of the K factor ($+4g$) and of the Grassmann measure (-4) . We consider functions of the type

$$F_{(1)^m(\dot{1})^m} = \sum_{n=0}^{\infty} \xi_{(1)^{m+n}(\dot{1})^{m+n}} (Y^{1\dot{1}})^n , \quad (8.8)$$

where

$$\xi_{(1)^p(\dot{1})^p} = \xi_{(a_1 \dots a_p)(\dot{a}_1 \dots \dot{a}_p)} v_1^{a_1} \dots v_1^{a_p} v_{\dot{1}}^{\dot{a}_1} \dots v_{\dot{1}}^{\dot{a}_p} \quad (8.9)$$

introduces a set of constant $SU(2)_L \times SU(2)_R$ multispinors, thus explicitly breaking the symmetry.

Let us examine the coupling (8.7) in some detail. First of all, from the term $(W^{1\dot{1}}W^{1\dot{1}})^{g+1}$ we only consider contributions of the type

$$(\theta^1)^4 (\bar{\theta}_2)^4 R^4 (T^{1\dot{1}})^m . \quad (8.10)$$

The Grassmann factor saturates the θ integrals. The harmonic dependence comes from the factor

$$(T^{1\dot{1}})^m = T^{(a_1(\dot{a}_1} \dots T^{a_m)\dot{a}_m)} v_{a_1}^1 \dots v_{a_m}^1 v_{\dot{a}_1}^{\dot{1}} \dots v_{\dot{a}_m}^{\dot{1}} . \quad (8.11)$$

Notice that the projection with commuting harmonic variables forces symmetrization of the indices of the T 's. Thus, this term contributes an irrep of each $SU(2)$ of weight m . Since the harmonic integral in (8.7) only sees the singlet part of the integrand, we have to find a matching irrep in the F sector, so that together they can form a singlet. Let us look at a term from (8.8) (where we replace the superfield Y by its first component ϕ),

$$\begin{aligned} \xi_{(1)^{m+n}(\dot{1})^{m+n}} (\phi^{1\dot{1}})^n &= \xi_{(a_1 \dots a_{m+n})(\dot{a}_1 \dots \dot{a}_{m+n})} v_1^{a_1} \dots v_1^{a_{m+n}} v_{\dot{1}}^{\dot{a}_1} \dots v_{\dot{1}}^{\dot{a}_{m+n}} \\ &\times \phi^{(b_1(\dot{b}_1} \dots \phi^{b_n)\dot{b}_n)} v_{b_1}^1 \dots v_{b_n}^1 v_{\dot{b}_1}^{\dot{1}} \dots v_{\dot{b}_n}^{\dot{1}} . \end{aligned} \quad (8.12)$$

The first factor involves only harmonics with upper $SU(2)$ indices, the second only with lower indices. Such products of harmonics are reducible. Using the defining conditions (8.1), we can decompose a reducible product of v_a^1 with v_1^b as follows: $v_a^1 v_1^b = 1/2 \delta_a^b + v_{\{a}^1 v_{1\}^b\}$, where $\{\}$ denotes the traceless part. Contracting the indices of all v^1 's and v_1 's with those of a subset of the v_1 's and v_1 's, we can eliminate the v^1 's and v_1 's from (8.12). The result is the irrep of weight m of each $SU(2)$ needed to match that in (8.11); any traceless combination $v_{\{i}^1 v_{1\}^j\}$ will contribute to an irrep of higher isospin without a match in (8.11), thus irrelevant for the harmonic integral. So, we can reduce (8.8) to its relevant part

$$\mathcal{F} = \sum_n \xi_{(a_1 \dots a_{m+n})(\dot{a}_1 \dots \dot{a}_{m+n})} v_1^{a_1} \dots v_1^{a_m} v_1^{\dot{a}_1} \dots v_1^{\dot{a}_m} \phi^{a_{m+1} \dot{a}_{m+1}} \dots \phi^{a_{m+n} \dot{a}_{m+n}} . \quad (8.13)$$

It is important to realize that the ξ tensor in (8.13) has all its indices symmetrized. This is the origin of the harmonicity constraint

$$\epsilon^{ab} \frac{\partial}{\partial v_1^a} \frac{\partial}{\partial \phi^{bb}} \mathcal{F} = 0 . \quad (8.14)$$

It involves a partial derivative with respect to v_1 . Strictly speaking, such an operation is illegal in the harmonic formalism, since the variables v_1 and v^1 are not independent, as can be seen from (8.1). However, in (8.13) there are only v_1 's left, so we can formally take such a derivative. In fact, if needed, (8.14) can also be expressed using covariant harmonic derivatives as in (2.32).

In principle, we could go on and discuss the coset space $SO(4, n)/SO(4) \times SO(n)$ parametrized by the scalars ϕ of the vector multiplets (8.4) in a manner similar to that of Sect. 3. The conclusion would be a second-order constraint analogous to (3.18). However, in six dimensions we do not have the setup of conformal supergravity of Sect. 4 which allowed us to fix the value of the charge Z_0 in (3.20). Therefore, we can make a prediction for the structure of this constraint, but we cannot explain the precise value of the coefficient obtained from the string calculation, see (8.31).

8.2 Decompactification of four-dimensional amplitudes

8.2.1 Decompactification limit

In order to round up the six-dimensional discussion, let us now check the field theory predictions by direct string calculations for the decompactification of the topological amplitude (5.2) from four to six dimensions, which corresponds to the coupling (8.7). Essentially, it was already shown in [5] that upon decomposing T^6 into $T^4 \times T^2$ and the subsequent reduction of the $\Gamma^{(6,22)}$ lattice into

$$\Gamma^{(6,22)} \rightarrow \Gamma^{(4,20)} \times \Gamma^{(2,2)} , \quad (8.15)$$

the weaker version of the first order harmonicity relation (6.14) is reduced to a relation for type II string theory compactified on $K3$, proved in [10]. Below, we will check the stronger relation (8.14) and compute its corresponding quantum anomaly.

In order to perform the reduction (8.15) we choose as in [5] P_{13}^L and its complex conjugate P_{24}^L of $\Gamma^{(6,22)}$ to be entirely in $\Gamma^{(2,2)}$ and the remaining four P_{12}^L , P_{14}^L and their complex conjugates P_{34}^L , P_{23}^L to form the $\Gamma^{(4,20)}$. In this way, the group $SU(4)$ is reduced to its subgroup $SU(2)_L \times SU(2)_R$ where $SU(2)_L$ and $SU(2)_R$ are acting on the indices (1, 3) and (2, 4), respectively. In the decompactification limit, P_{13} and P_{24} decouple and are dropped from the correlation function. In this way, $\Gamma^{(4,20)}$ lattice vectors are denoted by:

$$P_{a\dot{b}}^{(4,20),L}, P_A^{(4,20),R} \quad \text{with } a, \dot{b} = 1, 2 \quad (8.16)$$

and the index of the right-moving lattice momenta takes now the values $A = 1, \dots, 20$. Moreover, the square of the left-moving momenta will be denoted by

$$(P^{(4,20),L})^2 = \frac{1}{2} P_{a_1 \dot{b}_1}^{(4,20),L} P_{a_2 \dot{b}_2}^{(4,20),L} \epsilon^{a_1 a_2} \epsilon^{\dot{b}_1 \dot{b}_2}. \quad (8.17)$$

In order to make contact with the six-dimensional harmonic coordinates introduced in Section 8.1 we can assemble part of the $SU(4)$ harmonics \bar{u}_{+1} and \bar{u}_{+2} into the harmonics of $SU(2)_L \times SU(2)_R$ with the identification

$$(\bar{u}_{+1}^i, \bar{u}_{-1}^i) \rightarrow (v_1^a, v_2^a), \quad (8.18)$$

$$(\bar{u}_{+2}^i, \bar{u}_{-2}^i) \rightarrow (v_1^{\dot{a}}, v_2^{\dot{a}}), \quad (8.19)$$

and we recall that the $SU(2)_L \times SU(2)_R$ harmonics satisfy the completeness condition (8.1) and the unit determinant condition (8.2). Finally, the 1-loop heterotic amplitude (5.2) was shown in [5] to take the following form after the decompactification of the T^2

$$\mathcal{F}_g^{\text{dec}} \sim \int \frac{d^2 \tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1}(\tau, \bar{\tau}) \sum_{(P^L, P^R) \in \Gamma^{(4,20)}} \left(v_1^a P_{a\dot{b}}^L v_{\dot{1}}^{\dot{b}} \right)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}, \quad (8.20)$$

where from now on, we will drop the (4, 20) superscript of the lattice momenta and for further convenience, we define the following shorthand notation

$$P_{i\dot{1}}^L \equiv \left(v_1^a P_{a\dot{b}}^L v_{\dot{1}}^{\dot{b}} \right), \quad (8.21)$$

similar to the four-dimensional definition (5.10).

8.2.2 Harmonicity relation

We now study the six-dimensional harmonicity relation (8.14):

$$\epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial v_{\dot{1}}^{\dot{a}}} D_{a\dot{b},A} \mathcal{F}_g^{\text{dec}}, \quad (8.22)$$

where the covariant derivative $D_{a\dot{b},A}$ is with respect to the moduli forming the $\Gamma^{(4,20)}$ lattice. We can again apply simple rules for the differentials acting on the lattice momenta, similar to (6.4):

$$D_{a\dot{a},A} P_{b\dot{b}}^L = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} P_A^R, \quad D_{a\dot{a},A} P_B^R = \frac{\delta_{AB}}{2} P_{a\dot{a}}^L. \quad (8.23)$$

The computation can then be performed in the same straight-forward manner as in the four-dimensional case

$$\begin{aligned}
E_1^{\text{dec}} &\equiv \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial v_1^{\dot{a}}} D_{ab,A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\
&= \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial v_1^{\dot{a}}} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} \left[(2g-2) v_1^c v_1^{\dot{c}} \epsilon_{ac} \epsilon_{\dot{b}\dot{c}} + \pi i (\tau - \bar{\tau}) P_{1\dot{1}}^L P_{ab}^L \right] \cdot \\
&\quad \cdot (P_{1\dot{1}}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}.
\end{aligned}$$

The derivative with respect to the harmonic variable yields

$$\begin{aligned}
E_1^{\text{dec}} &= \epsilon^{\dot{a}\dot{b}} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} \left\{ \left[(2g-2) v_1^c \epsilon_{ac} \epsilon_{\dot{b}\dot{a}} - 2\pi \tau_2 v_1^c P_{ab}^L P_{c\dot{a}}^L \right] (P_{1\dot{1}}^L)^{2g-3} + \right. \\
&\quad \left. + (2g-3) \left[(2g-2) v_1^c v_1^{\dot{c}} \epsilon_{ac} \epsilon_{\dot{b}\dot{c}} - 2\pi \tau_2 P_{ab}^L (P_{1\dot{1}}^L) \right] v_1^c P_{c\dot{a}}^L (P_{1\dot{1}}^L)^{2g-4} \right\} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\
&= -(2g-2) \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-3} \left[(2g-1) \epsilon_{ac} v_1^c + 2\pi \tau_2 v_1^c P_{c\dot{a}}^L \epsilon^{\dot{a}\dot{b}} P_{ab}^L \right] \cdot \\
&\quad \cdot P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}.
\end{aligned}$$

We can now make use of the identity

$$P_{b\dot{a}}^L \epsilon^{\dot{a}\dot{b}} P_{ab}^L = -(P^L)^2 \epsilon_{ab}, \quad (8.24)$$

which simplifies the expression to

$$\begin{aligned}
E_1^{\text{dec}} &= \\
&- (2g-2) \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} \epsilon_{ac} v_1^c (P_{1\dot{1}}^L)^{2g-3} \left[(2g-1) - 2\pi \tau_2 (P^L)^2 \right] P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}
\end{aligned}$$

The special form of this term allows for the following rewriting

$$E_1^{\text{dec}} = -2i(2g-2) \int \frac{d^2\tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \epsilon_{ac} v_1^c \frac{\partial}{\partial \tau} \left[\tau_2^{2g-1} (P_{1\dot{1}}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right]$$

while a partial integration in τ finally yields

$$E_1^{\text{dec}} = (2g-2)\pi \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-3} G_g \sum_{(P^L, P^R)} \epsilon_{ac} v_1^c (P_{1\dot{1}}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}, \quad (8.25)$$

where we have once more made use of (5.7). We now confront this result with the following expression

$$v_1^a v_1^{\dot{a}} D_{a\dot{a},A} \mathcal{F}_{g-1}^{\text{dec}} \equiv D_{11,A} \mathcal{F}_{g-1}^{\text{dec}}, \quad (8.26)$$

which can be evaluated exactly in the same way as (8.22)

$$\begin{aligned} D_{1i,A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-4} G_g(\tau, \bar{\tau}) \sum_{(P^L, P^R)} (P_{1i}^L)^{2g-4} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\ = -2\pi \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-3} G_g \sum_{(P^L, P^R)} (P_{1i}^L)^{2g-3} P_A^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \end{aligned}$$

Comparing this expression to (8.25), we conclude

$$\epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial v_1^{\dot{a}}} D_{ab,A} \mathcal{F}_g^{\text{dec}} = -\frac{1}{2}(2g-2) \epsilon_{ab} v_1^b D_{1i,A} \mathcal{F}_{g-1}^{\text{dec}}. \quad (8.27)$$

Finally replacing the differentiation with respect to $v_1^{\dot{a}}$ by one with respect to v_1^a , we can derive a similar equation

$$\epsilon^{ab} \frac{\partial}{\partial v_1^a} D_{b\dot{a},A} \mathcal{F}_g^{\text{dec}} = -\frac{1}{2}(2g-2) \epsilon_{\dot{a}\dot{b}} v_1^{\dot{b}} D_{1i,A} \mathcal{F}_{g-1}^{\text{dec}}. \quad (8.28)$$

For both equations (8.27) and (8.28), the same considerations as in the four-dimensional case imply that the right hand side can be interpreted as an anomaly.

Notice that the left-hand side of (8.27) and (8.28) is exactly the harmonicity condition first derived in [6]. There, however, corrections to the equation by boundary terms of the Riemann surface as well as by certain contact terms in operator product expansions were neglected. In the later work [10], the missing of these extra contributions was pointed out and it was suggested that an additional contraction with harmonic coordinates would project out all extra terms. This was demonstrated by a careful analysis in the topological twisted theory. Indeed, if we project the free indices of (8.27) and (8.28) with v_1 and v_i respectively, we find

$$\begin{aligned} v_1^a \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial v_1^{\dot{a}}} D_{ab,A} \mathcal{F}_g^{\text{dec}} &= -\frac{1}{2}(2g-2) v_1^a \epsilon_{ab} v_1^b D_{1i,A} \mathcal{F}_{g-1}^{\text{dec}} = 0, \\ v_1^{\dot{a}} \epsilon^{ab} \frac{\partial}{\partial v_1^a} D_{b\dot{a},A} \mathcal{F}_g^{\text{dec}} &= -\frac{1}{2}(2g-2) v_1^{\dot{a}} \epsilon_{\dot{a}\dot{b}} v_1^{\dot{b}} D_{1i,A} \mathcal{F}_{g-1}^{\text{dec}} = 0, \end{aligned}$$

in complete agreement with [10], serving as an additional check for our computation.

8.2.3 Second order relation

Finally, we can also study the decompactification limit of the second order constraint (7.7), whose left-hand side becomes the following differential operator:

$$\epsilon^{ab} \epsilon^{\dot{a}\dot{b}} D_{a\dot{a},A} D_{b\dot{b},B} \mathcal{F}_g^{\text{dec}}. \quad (8.29)$$

Using again the differentiation rules (8.23), we can evaluate (8.29) in a straight-forward way

$$\begin{aligned}
E_2^{\text{dec}} &\equiv \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} D_{a\dot{a},A} D_{b\dot{b},B} \mathcal{F}_g^{\text{dec}} = \\
&= D_{a\dot{a},A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-3} \left[(2g-2) v_1^a v_1^{\dot{a}} - 2\pi\tau_2 \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} (P_{1\dot{1}}^L) P_{b\dot{b}}^L \right] \cdot \\
&\quad \cdot P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\
&= -4\pi \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} [2g\tau_2 - 2\pi\tau_2^2 (P^L)^2] P_A^R P_B^R (P_{1\dot{1}}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} + \\
&\quad + \delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_{g+1} \sum_{(P^L, P^R)} [(2g-1) - 2\pi\tau_2 (P^L)^2] (P_{1\dot{1}}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
&\quad - g\delta_{AB} \mathcal{F}_g^{\text{dec}}.
\end{aligned}$$

Following the same steps as before, we can re-write the first two lines as total derivatives with respect to τ , namely

$$\begin{aligned}
E_2^{\text{dec}} &= 8\pi i \int \frac{d^2\tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \frac{\partial}{\partial\tau} \left[\tau_2^{2g} (P_{1\dot{1}}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right] + \\
&\quad + 2i\delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} G_{g+1} \sum_{(P^L, P^R)} \frac{\partial}{\partial\tau} \left[\tau_2^{2g-1} (P_{1\dot{1}}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} \right] - \\
&\quad - g\delta_{AB} \mathcal{F}_g^{\text{dec}},
\end{aligned}$$

which after a partial integration become

$$\begin{aligned}
E_2^{\text{dec}} &= 4\pi^2 \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-2} P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
&\quad - \pi\delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-3} G_g \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-2} q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - g\delta_{AB} \mathcal{F}_g^{\text{dec}}.
\end{aligned}$$

In order to simplify this result, we also evaluate the expression

$$\begin{aligned}
&D_{1\dot{1},A} D_{1\dot{1},B} \mathcal{F}_{g-1}^{\text{dec}} = \\
&= -2\pi D_{1\dot{1},A} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-3} G_g \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-3} P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} = \\
&= 4\pi^2 \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-2} G_g \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-2} P_{a\dot{a}}^L P_A^R P_B^R q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2} - \\
&\quad - \pi\delta_{AB} \int \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2^{2g-3} G_g \sum_{(P^L, P^R)} (P_{1\dot{1}}^L)^{2g-2} P_{a\dot{a}}^L q^{\frac{1}{2}(P^L)^2} \bar{q}^{\frac{1}{2}(P^R)^2}. \tag{8.30}
\end{aligned}$$

We can thus obtain the relation

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}D_{a\dot{a},A}D_{b\dot{b},B}\mathcal{F}_g^{\text{dec}} = D_{1i,A}D_{1i,B}\mathcal{F}_{g-1}^{\text{dec}} - g\delta_{AB}\mathcal{F}_g^{\text{dec}}. \quad (8.31)$$

As already mentioned in Section 8.1, the general structure of this equation can be anticipated from field theory, especially, the existence of the term proportional to δ_{AB} on the right hand side of (8.31). However, due to the lack of the setup of conformal supergravity, we are not in a position to predict the exact coefficient $-g$, which is also different from the coefficient in the four-dimensional analog of the second order constraint (7.7).

Note finally, that the six-dimensional couplings $\mathcal{F}_g^{\text{dec}}$ (8.20) of the 1/2-BPS effective operator (8.7), although obtained by taking the decompactification limit of the $\mathcal{N} = 4$ topological amplitudes $\mathcal{F}_g^{(3)}$, they are not given by the topological theory on $K3$. The reason is that in their exact genus $g + 1$ type II expression, the $\det \text{Im}\tau$ factors from the space-time coordinates do not cancel. Thus, these couplings are semi-topological, in the sense that string oscillator modes do not contribute, and upon compactification on a T^2 they become exactly topological.

9 Conclusions

In conclusion, in this work, we generalized the holomorphicity property of the $\mathcal{N} = 2$ supersymmetric couplings involving vector multiplets to the moduli dependence of the $\mathcal{N} = 4$ couplings of 1/2-BPS operators defined in harmonic superspace. An example of such operators is provided by the two series found in [5], involving powers of the (superdescendant of the) $\mathcal{N} = 4$ chiral Weyl superfield K whose coupling-coefficients are functions of the $\mathcal{N} = 4$ vector moduli and are computed by the $\mathcal{N} = 4$ topological string on $K3 \times T^2$.

The resulting harmonicity or analyticity property is expressed in terms of two sets of differential constraints: the first requires the vanishing of one scalar and one harmonic derivatives, while the second imposes two scalar (covariant) derivatives to give back the same coupling up to a multiplicative constant proportional to its (super)conformal weight. We verified these equations on the string side using the explicit expressions for the couplings of one of the two series as 1-loop heterotic integrals on T^6 .

We also extended the above analysis to $\mathcal{N} = 2$ 1/2-BPS terms in six dimensions and we checked the resulting equations for the couplings obtained in the decompactification limit of the four-dimensional $\mathcal{N} = 4$ topological amplitudes considered before. In principle, our analysis can be generalized in a straight-forward way to the couplings of any 1/2-BPS operator of extended supersymmetry in any space-time dimension.

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